THE PRIMORDIAL GRAVITATIONAL WAVE BACKGROUND IN STRING COSMOLOGY

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Abstract

We find the spectrum $P(\omega)d\omega$ of the gravitational wave background produced in the early universe in string theory. We work in the framework of the recently discussed String Driven Cosmology, whose scale factors are computed with the low-energy effective string equations as well as selfconsistent solutions of General Relativity with a gas of strings as source. The scale factor evolution is described by an early string driven inflationary stage with an instantaneous transition to a radiation dominated stage and successive matter dominated stage. This is a string cosmology expanding evolution always running on positive proper cosmic time.

The spectrum of the generated gravitons is computed in the framework of Quantum Field Theory and in the appropiated effective String Cosmology context. We study and show explicitly the effect of the dilaton field, characteristic to this kind of cosmologies. We compute the spectrum for the same evolution description with three differents approachs. Some features of gravitational wave spectra, as peaks and asymptotic behaviours, are found direct consequences of the dilaton involved in the string low energy effective action and not only of the scale factor evolution.

We make use of a careful treatment of the scale factor evolution and involved transitions. This allows us to compute a full prediction on the power spectrum of gravitational waves without any free-parameters. A comparative analysis of different treatments, solutions and compatibility with observational bounds or detection perspectives is made.

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1 Introduction and Results

In the latest years, an impressive scientific effort has been made on the subject of production of relic gravitational waves. The spectrum of relic gravitational waves constitutes an important source of information about the conditions and dynamics of the early Universe. From the pioneering works relating the very early Universe with the spectrum of metric fluctuations ([1],[2],[3]) a lot of work has followed computing the gravitational waves background in different evolution models (see for instance [4],[5],[6],[7],[8],[9]). The relic gravitational wave background is generated as a consequence of the amplification by the Universe expansion of the graviton fields, which decouples from matter at very early stages. Therefore, their spectrum is defined by conditions in the very early epoch and remains practically unaffected until our days, since the medium is transparent to their propagation.

In addition, important improvements on the projects for direct detection of gravitational waves, as well as on the measurements of astrophysical phenomena affected by them have been performed or are expected for the next decade. This will make possible to extract constraints and to put upper limits on the magnitude of the generation and amplification of gravitational waves and thereafter, on the particular model of evolution considered for the universe. Currently, the search of direct measurements by means of resonant bar detectors (e.g., NAUTILUS, AURIGA) has been followed by the development of laser interferometers (LIGO, LISA). The sensitivities to be reached for the next generation of laser interferometers detectors could fix these constraints on ranges still more interesting from the point of view of early universe theories ([10]). Indirect constraints can be obtained from the closure relation, nucleosynthesis bound, pulsar timing measurements and isotropy of Cosmic Microwave Background Radiation (CMB) ([11]). Finally, there would be imprints on the CMB temperature anisotropy spectrum through Sachs-Wolfe effect ([12]) and polarization fluctuations originated by scattering Thompson on a tensorially perturbed metric ([13]).

This study is particularly interesting from the point of view of String Theory in Cosmology since gravitational waves are one of the observable tracks to be extracted from the fundamental theory. The decoupling of gravitons from another kind of matter has taken place very close to the Planck Energy Scale. Thus, the shape of the spectrum could carry information about the physics on such energy and lenght scales. A number of studies computing gravitational wave production in the context of string cosmology scenarii has been made in the last years. ([14],[15],[16],[17]). Oughted to the intrinsic uncertainty in the cosmological model, the spectra computed in these works depend on free parameters coming from the earliest stages of Universe, far from observational knowledge. No firm free-parameter prediction is extracted, although the existence of a peak and end-point in the string cosmology gravitational wave background has been claimed ([18]).

In this paper, we study the production of a primordial stochastic gravitational wave background generated in the context of selfconsistent string cosmology ([19],[20],[21],[22]). From both treatments therein explained, low energy effective string action and selfconsistent solutions of General Relativity plus string matter, we have obtained a minimal model for the evolution of the scale factor [23]. The String Driven Cosmological Background is characterized by an inflationary inverse power stage $a_I(t) \sim (t_I - t)^{-\frac{2}{d+1}}$ plus a radiation dominated stage $a_{II}(t) \sim (t - t_{II})^{\frac{2}{d+1}}$ and a matter dominated stage $a_{III}(t) \sim t^{\frac{2}{d}}$. The cosmological model is linked to the Universe observational information by means of descriptive temporal variables as performed in [24] and [23]. Our results are thus expressed in terms of standard times for inflation-radiation dominated transition $\mathcal{T}_r$, and radiation dominated-matter dominated transition $\mathcal{T}_m$. The conformal time variable η has been defined with totally satisfactory continuity conditions in the transitions.

We compute exact expressions for the power spectrum $P(\omega)d\omega$ and contribution to the energy density Ω_{GW} of primordial gravitational waves generated in the transition among the inflationary and radiation dominated stages. These expressions are fully predictive and free-parameter, since the cosmological model has been constrainted with minimal and well stablished observational Universe information (the transition times $\mathcal{T}_r$ and $\mathcal{T}_m$). Having fixed a minimal but satisfactory model for the scale factor, we study the production of gravitational waves by making three approaches in the perturbation treatment.

The role of the dilaton, a characteristic field in low energy effective string theory and present in our model, is proved to be crucial since gives a strong signature on the shape and magnitude of these spectra. The computations of graviton perturbations with no dilaton role are showed to be equivalent to standard Quantum Field Theory computations, like those reported in Allen [4] and Grishchuk [5], but notorious differences are found in the spectra oughted to the particular character of the inflationary string driven scale factor, an inverse power type. We obtain an exact expression for the power spectrum and energy density contribution (eqs.(5.38) and (5.45)) in terms of Hankel functions. The power spectrum asymptotic behaviours at low and high frequencies are both vanishing with dependences $\omega^{\frac{1}{3}}$ and ω^{-1} respectively. This gives a gravitational wave contribution to the energy density asymptotically constant at high frequencies on $\Omega_{GW} \sim 10^{-26}$. The slope change produces a peak in the power spectra around a characteristic frequency totally determined by the model of value $\omega_x \sim 1.48$ Mhz.

Computations with partial dilaton role, considering this in the perturbation equation but not on the perturbation itself, show some differences with the last ones, althought the general characteristics are very similar. The formal expressions for the power spectrum and energy density contribution are the same of the No Dilaton case (eqs.(5.38) and (5.45)), but they involve Hankel functions with different orders. Both asymptotic regimes for small and high frequencies vanish again, but now with dependences $\omega^{\frac{5}{3}}$ and ω^{-1} respectively. The peak appears around the same characteristic frequency, but its value is one order of magnitude lower than in the No Dilaton case, as well as the asymptotic constant contribution to the energy density Ω_{GW}.

In the opposite case, when the full dilaton role is accounted, general characteristics as well as orders of magnitude of the graviton spectrum are drastically modified. The formal expressions are different (eqs.(5.83),(5.84)). We extract the asymptotic behaviours too: for low frequencies, the dependence is vanishing again with $\omega^{\frac{5}{3}}$. For high frequencies, an asymptotic divergent behaviour proportional to ω is found. Similarly, the contribution to the critical energy density in gravitational waves is divergent at high frequencies, as ω^2. The change of slope is less visible and no clear peaks are found. The transition from the low frequency to the high frequency regime is slower than in the previous cases (with no dilaton or partial dilaton roles) and full analytical expressions are needed on a wider range $10^6 \sim 10^9 Hz$. Comparaison of these summarized results can be found in Tables (3), (4) and (5) and in Figures (1) and (2).

The behaviour obtained in the full dilaton treatment is a common characteristic to the String Cosmology low energy effective treatments. We conclude that the Brans-Dicke frame (dilaton field) involved and the type of scale factor predicted (inverse power type) are responsible of these features. More conclusions (the dilaton role, string and no-string cosmologies) are given in Section VII.

In Section II, we recall the derivation of the background solutions obtained in String Cosmology, the richness of String Theory applied to cosmology is visible along this section. In Section III we summarize the String Driven Cosmological Background in order to obtain an approximative but suitable description of the cosmology of our observational Universe. We translate this description to a conformal time-type variable more convenient for further work. Section IV shows the computation of gravitational wave production in the framework of Quantum Field Theory and shows how the dilaton role can be considered. In Section V we particularize to our cosmological description and obtain exact expressions in suitable measurement units both for power spectrum and energy density in each one of the three treatments. In Section VI we analyze the asymptotic behaviours. Finally, we elaborate and discuss our results and compare to other String Cosmology computations as well as to pure General Relativity gravitational wave computations, and present our conclusions in Section VII.

2 String Driven Cosmological Background

The String Driven Cosmological Background is a minimal model of the Universe evolution totally extracted from effective String Theory. Details on the model can be found in Ref.[23], but here we summarize it. Two ways allowing extraction of cosmological descriptions for the background from string theory have been used. One is the low energy effective string equations plus string action matter. Solutions of these equations are an inflationary inverse power law evolution of the scale factor, as well as a radiation dominated behaviour. On the other hand, selfconsistent Einstein equations with a classical gas of strings as sources will give us again a radiation dominated behaviour and a matter dominated description.

In this whole and next sections, unless opposite indication, the metric is defined in lenght units. Thus, the (0,0) component is always time coordinate $\mathcal{T}$ multiplied by constant c, that means $t = c\mathcal{T}$. Derivatives are taken with respect to this coordinate t. Otherwise, constants c, $\hbar$ and G are explicitly showed. We consider a spatially flat, homogeneous and isotropic background and we write the metric in synchronous frame ($g_{00} = 1$, $g_{0i} = 0 = g_{0a}$) as:

$$g_{\mu\nu} = \mathrm{diag}(1, -a^2(t)\ \delta_{ij}) \tag{2.1}$$

The earliest stages in our model are provided by the low energy effective context. The scale factor is extracted by extremizing with respect to dilaton field, graviton field and matter sources the low energy string effective action (to the lowest order in expansion of powers of α'), which in the Brans-Dicke or string frame can be written as [25],[19],[14],[21]:

$$S = -\frac{c^3}{16\pi G_D} \int d^{d+1}x \sqrt{\mid g \mid} e^{-\phi} \left(R + \partial_\mu \phi \partial^\mu \phi - \frac{H^2}{12} + V \right) + S_M \tag{2.2}$$

where S_M is the corresponding action for the strings as matter sources, $H = dB$ is the antisymmetric tensor field strenght and V is related with dilaton potential and vanishing for some critical dimension. The dilaton field ϕ depends explicitly only upon time coordinate. D is the total spacetime dimension, $D = d + 1$ where d is the number of spatial dimensions. The string matter is included as a classical source which stress energy tensor in the perfect fluid approximation takes the form:

$$T_\mu{}^\nu = \mathrm{diag}(\rho(t), -P(t)\delta_i{}^j) \tag{2.3}$$

where ρ is the energy density and P the pressure for the matter sources. Here, we will consider antisymmetric tensor $H_{\mu\alpha\beta}$ as well as the potential B vanishing. We do not consider here the effects oughted to a non-vanishing dilaton potential. Thus, the low energy effective equations are obtained:

$$
\begin{aligned}
\dot{\bar{\phi}}^2 - 2\ddot{\bar{\phi}} + dH^2 &= 0 \\
\dot{\bar{\phi}}^2 - dH^2 &= \frac{16\pi G_D}{c^4}\, \bar{\rho}\, e^{\bar{\phi}} \\
2(\dot{H} - H\dot{\bar{\phi}}) &= \frac{16\pi G_D}{c^4}\, \bar{p}\, e^{\bar{\phi}}
\end{aligned}
\tag{2.4}
$$

where $H = \frac{\dot{a}}{a}$ and we use the shifted string duality invariant expressions for the dilaton $\bar{\phi} = \phi - \ln \sqrt{|\, g \,|}$, matter energy density $\bar{\rho} = \rho a^d$ and pressure $\bar{p} = P a^d$.

The inflationary String Driven stage appears as a new selfconsistent solution of the low energy effective equations (2.4) sustained by a gas of stretched or unstable string sources ([19],[20]). The pressure and energy density of this string behaviour in curved background satisfy the equation of state $P = \gamma\rho$ as matter sources, where $d\ \gamma = -1$ for the unstable behaviour in the metric here considered. Thus, the equation of state to be considered for the string sources is given by:

$$
P = -\frac{1}{d}\,\rho \tag{2.5}
$$

With this, the selfconsistent solution of effective equations (2.4) is:

$$
\begin{aligned}
& a(t) = A_I (t_I - t)^{-Q} \qquad 0 < t < t_r < t_I \quad , \quad Q = \frac{2}{d+1} \\
& \phi(t) = \phi_I + 2d \ln a(t) \\
& \rho(t) = \rho_I (a(t))^{(1-d)} \\
& P(t) = -\frac{1}{d}\,\rho(t) \quad = \quad -\frac{\rho_I}{d}(a(t))^{(1-d)}
\end{aligned}
\tag{2.6}
$$

Notice here t is the proper cosmic time coordinate, running on positive values; d is the number of expanding spatial dimensions; ρ_I, ϕ_I are integration constants and A_I, t_I parameters to be fixed by the further evolution of scale factor, the parameter t_I being greater than the exit of the inflationary stage t_r.

It is convenient for further work to express the solutions in conformal time η such $d\eta = \frac{dt}{a(t)}$:

$$
\begin{aligned}
& a_{inf}(\eta) \sim (\eta_I - \eta)^q \quad , \qquad q = -\frac{2}{d+3} \\
& \phi_{inf}(\eta) \sim -\frac{4d}{d+3}\ln(\eta_I - \eta)
\end{aligned}
\tag{2.7}
$$

where η_I is a parameter to be determined in the matching with the next cosmological stage.

As discussed in ref.[23], this is a new string cosmology solution, obtained without exploting manifestly duality relations among the solutions, and describing an accelerated expansion running on the positive branch of cosmic temporal coordinate. In this way, this solution is different to the "Pre-Big Bang" solutions found in literature [14].

On the other hand, the dual to unstable string behaviour in curved backgrounds follows a typical radiation type equation of state $P = \frac{1}{d}\,\rho$ [19]. The effective equations (2.4) plus this equation of state give the following behaviour describing a radiation dominated stage [19], [14]:

$$\begin{aligned} a(t) &= A_{II}\, t^R \qquad t_r < t \quad , \qquad R = \frac{2}{d+1} \\ \phi(t) &= \phi_{II} \\ \rho(t) &= \rho_{II}(a(t))^{-(1+d)} \\ P(t) &= \frac{1}{d}\,\rho(t) = \frac{\rho_{II}}{d}(a(t))^{-(1+d)} \end{aligned} \tag{2.8}$$

here ϕ_{II}, ρ_{II} are integration constants, and A_{II} a parameter to be fixed by the evolution of scale factor. Notice the dilaton remains "frozen" at constant value. Again, we define the corresponding conformal time to this stage. The solution takes the form:

$$\begin{aligned} a_{rad}(\eta) &\sim (\eta - \eta_{II})^r \quad , \quad r = \frac{2}{d-1} \\ \phi_{rad}(\eta) &= \phi_{II} \end{aligned} \tag{2.9}$$

where η_{II}, ϕ_{II} are constants to be determined with the scale factor evolution.

Cosmological backgrounds can be found too, as selfconsistent solutions of the General Relativity Einstein equations selfsustained by the string sources evolving in them, as developped in [19],[20],[21]. In the curved backgrounds here considered, the Einstein equations take the form:

$$\begin{aligned} \frac{1}{2}d(d-1)H^2 &= \rho \\ (d-1)\dot{H} + P + \rho &= 0 \end{aligned} \tag{2.10}$$

and the matter source is described by a gas of non interacting classical strings, whose equation of state includes the different possible behaviours of strings in curved spacetimes: unstable, dual to unstable and stable, each one with string densities $\mathcal{U}$, $\mathcal{D}$ and $\mathcal{S}$ respectively. Taking into account the properties of each behaviour (see [19]), the energy density and the pressure of the string gas are described by:

$$\rho = \frac{1}{(a(t))^d}\left(\mathcal{U}a(t) + \frac{\mathcal{D}}{a(t)} + \mathcal{S}\right) \tag{2.11}$$

$$P = \frac{1}{d}\,\frac{1}{(a(t))^d}\left(\frac{\mathcal{D}}{a(t)} - \mathcal{U}a(t)\right) \tag{2.12}$$

Equations (2.11) and (2.12) are qualitatively corrects for every t and become exacts in the asymptotic regimes, leading to obtain the radiation dominated behaviour of the scale factor, as well as the matter dominated behaviour. The first one is obtained in the limit $a(t) \to 0$ and $t \to 0$, where the dual to unstable behaviour dominates in the equations (2.11) and (2.12) and gives us:

$$\begin{aligned} \rho(t) &\sim \mathcal{D}\,(a(t))^{-(d+1)} \\ P(t) &\sim \frac{1}{d}\,\mathcal{D}\,(a(t))^{-(d+1)} \end{aligned} \tag{2.13}$$

Solving the Einstein equations (2.10) with sources following eqs.(2.13), yields the scale factor solution:

$$a(t) \sim \left(\frac{2\mathcal{D}}{d(d-1)}\right)^{\frac{1}{d+1}} \left(\frac{d+1}{2}\right)^{\frac{2}{d+1}} (t - t_{II})^{R} \quad , \quad R = \frac{2}{d+1} \tag{2.14}$$

$a(t)$ describes the evolution of a FRW radiation dominated stage, here the parameter t_{II} will be fixed by the further evolution of the scale factor. This is coherent with the fact that dual to unstable strings behave in a similar way to massless particles, i.e. radiation [19]. In conformal time and rearranging constants, this solution reads:

$$a_{rad}(\eta) = \alpha_{II}(\eta - \eta_{II})^{r} \quad , \quad r = \frac{2}{d-1}$$

Notice that the scale factor now described is equivalent to the before obtained in low energy effective treatment, eq.(2.9).

In the opposite limit $a(t) \to \infty$, $t \to \infty$, the unstable density $\mathcal{U} \to 0$ and the stable behaviour $\mathcal{S}$ becomes dominant. The equation of state reduces to:

$$\begin{aligned} \rho &\sim \mathcal{S}\,(a(t))^{-d} \\ P &= 0 \end{aligned} \tag{2.15}$$

and from solving eqs.(2.10) with eqs.(2.15), the solution of a matter dominated stage emerges (Notice that stable strings behave as cold matter):

$$a(t) \sim \left(\frac{d}{(d-1)}\frac{\mathcal{S}}{2}\right)^{\frac{1}{d}} (t - t_{III})^{M} \quad , \quad M = \frac{2}{d} \tag{2.16}$$

here t_{III} is fixed by the whole scale factor evolution. This scale factor in conformal time takes the form:

$$a_{mat}(\eta) = \alpha_{III}(\eta - \eta_{III})^{m} \quad , \quad m = \frac{2}{d-2} \tag{2.17}$$

here the constants α_{III} and η_{III} will be determined from the transition between radiation and matter dominated stages.

The evolution model with the String Driven inflationary stage eq.(2.6), followed by a radiation dominated stage (eqs.(2.8) or (2.14) are equivalents for our purposes) and a matter dominated stage eq.(2.16) is the String Driven Cosmological Background that we will study.

Notice that we consider the dilaton field remaining practically constant and vanishing from the exit of inflation, as suggested in the String Driven Radiation Dominated Solution, until the current time. Notice also the Brans-Dicke frame for the metric-dilaton coupling where the string action has been written eq.(2.2) and both an inflationary and a radiation dominated stages have been extracted. The last one can be obtained too selfconsistently from the Einstein equations plus string matter, the same treatment where the matter dominated current stage is obtained too. In the table (1) the scale factor, dilaton and equation of state obtained from these treatments can be observed in a comparative way.

	String Driven Solutions		*String Sources in G.R.*	
	LEE + ($\mathcal{U}$,$\mathcal{D}$) strings		EE + gas ($\mathcal{U}$,$\mathcal{D}$,$\mathcal{S}$) strings	
	inflation	radiation	radiation	matter
$a(t) \sim$	$(t_I - t)^{-\frac{2}{d+1}}$	$t^{\frac{2}{d+1}}$	$t^{\frac{2}{d+1}}$	$t^{\frac{2}{d}}$
$a(\eta) \sim$	$(\eta_I - \eta)^{-\frac{2}{d+3}}$	$(\eta - \eta_{II})^{\frac{2}{d-1}}$	$(\eta - \eta_{II})^{\frac{2}{d-1}}$	$(\eta - \eta_{III})^{\frac{2}{d-2}}$
$\phi(t) \sim$	$\left(-\frac{4d}{d+1}\right) \ln(t_I - t)$	cte	- -	- -
$\phi(\eta) \sim$	$\left(-\frac{4d}{d+3}\right) \ln(\eta_I - \eta)$	cte	- -	- -
$\rho(t) \sim$	$(a(t))^{(1-d)}$	$(a(t))^{-(1+d)}$	$\mathcal{D}(a(t))^{-(1+d)}$	$\mathcal{S}(a(t))^{-d}$
string	*unstable*	*dual to u.*	*dual to u.*	*stable*
sources	$P = -\frac{1}{d}\rho$	$P = \frac{1}{d}\rho$	$P = \frac{1}{d}\rho$	$P = 0$

Table 1: Selfconsistent solutions giving the String Driven Cosmological Background

Table comparative of the solutions for the scale factor $a(t)$, $a(\eta)$. t is cosmic time, η is conformal time and ϕ is the dilaton field. *LEE* means low energy string effective equations in the Brans-Dicke frame, for a $(d+1)$ homogeneus, isotropic and spatially flat background; antisymmetric tensors and dilaton potentials are neglected. *EE* means the General Relativity Einstein Equations plus a gas of strings as classical sources matter. The dominant string behaviour as matter sources and their corresponding equation of state are written in the last row.

3 Scale Factor Description

Since the behaviours above extracted are asymptotic results, it is not possible to give here the detail of transitions among the different stages. It would be expected some changes in the regimes or equations governing each stage and leading to the next one, but this is an open question in the framework of string theory both for inflation-radiation dominated as well as radiation dominated-matter dominated transition. Taking the simplest option, we consider the "real" scale factor evolution minimally described as:

$$
\begin{aligned}
a_I(t) &= A_I(t_I - t)^{-Q} \qquad && t \in (t_i, t_r) \\
a_{II}(t) &= A_{II}\, t^R && t \in (t_r, t_m) \\
a_{III}(t) &= A_{III}\, t^M && t \in (t_m, t_0)
\end{aligned}
\tag{3.1}
$$

with transitions at least not excessively long at t_r beginning of radiation dominated stage and t_m beginning of matter dominated stage. We define also a beginning of inflation at t_i and a current time t_0.

From the point of view of String Theory applied to Cosmology, it would be reasonable to have not instantaneous and continuous transitions at t_r and t_m [23]. But this condition greatly difficults the computation that we want make because it introduces free parameters, loss of physical meaning and effects of an unknown transition dynamics [4] oughted to discontinuity on the scale factor. In order to construct a suitable minimal model for gravitational wave computations, we will merge our lack of knowledge on real transitions by means of descriptive temporal variables in function of which the modelized transitions at $\bar{t_1}$ and $\bar{t_2}$ are instantaneous and continuous.

We link this descriptive scale factor with the minimal information about the observational Universe. We consider the standard values for cosmological times: the radiation-matter transition held about $\mathcal{T}_m \sim 10^{12}s$, the beginning of radiation stage at $\mathcal{T}_r \sim 10^{-32}s$ and the current age of the Universe $\mathcal{T}_0 \sim {H_0}^{-1} \sim 10^{17}s$ (The exact numerical value of $\mathcal{T}_0$ turns out not crucial here). We impose also to our description to satisfy the same scale factor expansion (or scale factor ratii) reached in each one of the three stages considered in the real model (3.1). Explicit computations can be found in [24] and leds finally to this scale factor written in cosmic time-type variables in the more convenient way for our further work:

$$
\begin{aligned}
\bar{\bar{a}}_I(\bar{\bar{t}}) &= \bar{\bar{A}}_I(\bar{\bar{t}}_I - \bar{\bar{t}})^{-Q} \qquad && \bar{\bar{t}}_i < \bar{\bar{t}} < \bar{t_1} \\
\bar{a}_{II}(\bar{t}) &= \bar{A}_{II}(\bar{t} - \bar{t}_{II})^R && \bar{t_1} < \bar{t} < \bar{t_2} \\
a_{III}(t) &= A_{III}(t)^M && \bar{t_2} < t < t_0
\end{aligned}
\tag{3.2}
$$

with continuous transitions at $\bar{t_1}$ and $\bar{t_2}$ for both the scale factor and first derivatives with respect to the descriptive cosmic time-type variables $\bar{\bar{t}}$, $\bar{t}$ and t. In terms of the standard observational times t_r and t_m, the transitions $\bar{t_1}$, $\bar{t_2}$ and the beginning of the inflationary stage description $\bar{\bar{t}}_i$ are expressed as:

$$
\begin{aligned}
\bar{t_1} &= \frac{R}{M} t_r + \left(1 - \frac{R}{M}\right) t_m \qquad , \qquad \bar{t_2} = t_m \\
\bar{\bar{t}}_i &= \left(\frac{R}{M} + \frac{Q}{M}\frac{t_r - t_i}{t_r - t_I}\right) t_r + \left(1 - \frac{R}{M}\right) t_m
\end{aligned}
\tag{3.3}
$$

and the set of parameters of the scale factor (3.2) can be written too in terms of t_r, t_m and the global scale factor $\bar{A_{II}}$:

$$
\begin{aligned}
\bar{\bar{t_I}} &= t_r\Big(\frac{R}{M}+\frac{Q}{M}\Big)+t_m\Big(1-\frac{R}{M}\Big) \quad , \quad \bar{t_{II}} = \Big(1-\frac{R}{M}\Big)t_m \\
\bar{\bar{A_I}} &= \bar{A_{II}}\Big(\frac{Q}{M}\Big)^Q\Big(\frac{R}{M}\Big)^R {t_r}^{R+Q} \quad , \quad A_{III} = \bar{A_{II}}\Big(\frac{R}{M}\Big)^R {t_m}^{R-M}
\end{aligned}
\tag{3.4}
$$

Notice that the intermediate descriptive scale factor could not be physical, but this is not important under the purposes here. Since the description is linked with observational information and it satisfies the proper scale factor ratii, our computation will be equivalent to that made on a full and physical model. These are equivalents unless the features left on gravitational wave production by the details of an in any case unknown dynamics of transition and minimized through their modelization in satisfactory continuos way, as made here. The time variables $\bar{\bar{t}}$ of inflationary stage, and $\bar{t}$ of radiation stage are not a priori exactly equal to the physical time coordinate at rest frame (multiplied by c), but transformations (dilatation plus translation) of it. The low energy effective action equations from where the scale factor, dilaton and equation of state have been extracted, allows this transformations. With this treatment of cosmological scale factor, we will attain computations free of "by hand" parameters, and with full predictibility as can be seen in the next sections.

It is also useful to express this model in conformal time variables η, defined as $d\eta = \frac{dt}{a(t)}$. We will have a conformal time variable for each stage ($\bar{\bar{\eta}}$, $\bar{\eta}$ and η) oughted to the different scale factor shape and different descriptive temporal variable on which we integrate. In this way, the conformal time variable constructed enjoys continuity on their derivatives along the transitions at $\bar{t_1}$ and $\bar{t_2}$ [24].

$$
\frac{d\bar{\bar{\eta}}}{d\bar{\bar{t}}}\Big|_{\bar{t_1}^-} = \frac{d\bar{\eta}}{d\bar{t}}\Big|_{\bar{t_1}^+} \quad , \quad \frac{d\bar{\eta}}{d\bar{t}}\Big|_{\bar{t_2}^-} = \frac{d\eta}{dt}\Big|_{\bar{t_2}^+}
$$

We can impose also continuity conditions in order to have a conformal time uniquely defined at transitions. $\eta_1 = \bar{\eta}(\bar{t_1}) \equiv \bar{\bar{\eta}}(\bar{t_1})$ and $\eta_2 = \eta(\bar{t_2}) \equiv \bar{\eta}(\bar{t_2})$ (In this respect, different ways can be found in literature, see [24]). We use the remaining freedom in put a simplicity condition, asking a simple conformal time dependence in radiation dominated stage. It yields the next expressions for the scale factor in conformal time:

$$
\begin{aligned}
a_I(\bar{\bar{\eta}}) &= \alpha_I(\eta_I-\bar{\bar{\eta}})^{-q} \qquad && \eta_i < \bar{\bar{\eta}} < \eta_1 \\
a_{II}(\bar{\eta}) &= \alpha_{II}(\bar{\eta})^r && \eta_1 < \bar{\eta} < \eta_2 \\
a_{III}(\eta) &= \alpha_{III}(\eta_{III}+\eta)^m && \eta_2 < \eta
\end{aligned}
\tag{3.5}
$$

with continuous and suddenly transitions (continuity of scale factor and first derivative with respect to conformal time) at η_1 and η_2. The six parameters satisfy these matching relations:

$$\begin{aligned}
\alpha_I &= \alpha_{II}\left(\frac{q}{r}\right)^q (\eta_1)^{r+q} \quad , \quad \alpha_{III} = \alpha_{II}\left(\frac{m}{r}\right)^{-m} (\eta_2)^{r-m} \\
\eta_I &= \eta_1\left(1+\frac{q}{r}\right) \quad , \quad \eta_{III} = \eta_2\left(\frac{m}{r}-1\right)
\end{aligned} \tag{3.6}$$

Because its construction on a cosmic time description with fully continuity conditions, all the parameters involved in this conformal time scale factor have a complete and unique expression as functions of the observational standard transition times t_r and t_m, beginning of inflationary stage t_i, the exponents Q, R,M of cosmic time dependences and the global scale factor constant $\bar{A_{II}}$:

$$\begin{aligned}
q &= \frac{Q}{Q+1} \quad , \quad r = \frac{R}{1-R} \quad , \quad m = \frac{M}{1-M} \\
\eta_I &= \frac{R+Q}{R(Q+1)(1-R)}\frac{(\frac{R}{M}t_r)^{1-R}}{\bar{A_{II}}} \\
\alpha_I &= \left(\bar{A_{II}}\left(\frac{R}{M}\right)^R\left(\frac{Q}{M}\right)^Q\frac{1}{(Q+1)^Q}\,{t_r}^{R+Q}\right)^{\frac{1}{Q+1}} \\
\alpha_{II} &= \bar{A_{II}}^{\frac{1}{1-R}}(1-R)^{\frac{R}{1-R}} \\
\eta_{III} &= \frac{M-R}{R(1-M)(1-R)}\frac{(\frac{R}{M}t_m)^{1-R}}{\bar{A_{II}}} \\
\alpha_{III} &= \left(\bar{A_{II}}(1-M)^M\left(\frac{R}{M}\right)^R {t_m}^{R-M}\right)^{\frac{1}{1-M}}
\end{aligned} \tag{3.7}$$

Meanwhile, the conformal time transitions η_1, η_2 and η_i have the expressions:

$$\begin{aligned}
\eta_1 &= \frac{\left(\frac{R}{M}\,t_r\right)^{1-R}}{(1-R)\bar{A_{II}}} \quad , \quad \eta_2 = \frac{\left(\frac{R}{M}\,t_m\right)^{1-R}}{(1-R)\bar{A_{II}}} \\
\eta_i &= \frac{\left(\frac{R}{M}\,t_r\right)^{1-R}}{(Q+1)\bar{A_{II}}}\left[\frac{R+Q}{R(1-R)} - \frac{Q}{R}\left(\frac{t_i-t_I}{t_r-t_I}\right)^{Q+1}\right]
\end{aligned} \tag{3.8}$$

It is also possible give expressions for the conformal time in each stage as function of the proper cosmic time:

$$\begin{aligned}
\bar{\bar{\eta}} &= \frac{(\frac{R}{M}t_r)^{1-R}}{\bar{A_{II}}(Q+1)}\left[\frac{R+Q}{R(1-R)}-\frac{Q}{R}\left(\frac{t_I-t}{t_I-t_r}\right)^{Q+1}\right] \quad t\in(t_i,t_r) \\
\bar{\eta} &= \frac{(\frac{R}{M}t)^{1-R}}{\bar{A_{II}}(1-R)} \quad t\in(t_r,t_m) \\
\eta &= \frac{(\frac{R}{M}t_m)^{1-R}}{\bar{A_{II}}(1-M)}\left[\frac{R-M}{R(1-R)}+\frac{M}{R}\left(\frac{t}{t_m}\right)^{1-M}\right] \quad t\in(t_m,t_0)
\end{aligned} \tag{3.9}$$

Finally, it is useful for further work to compute here the value for the conformal time and scale factor at current time η_0. If we define $t_0 = c\mathcal{T}_0$ and from eqs.(3.9), it is easily seen:

$$\eta_0 = \frac{(\frac{R}{M}t_m)^{1-R}}{\bar{A}_{II}(1-M)}\left[\frac{R-M}{R(1-R)} + \frac{M}{R}\left(\frac{t_0}{t_m}\right)^{1-M}\right] \tag{3.10}$$

The scale factor written in conformal time for current time takes the form $a_{III}(\eta_0) = \alpha_{III}(\eta_{III} + \eta_0)^m$. From eqs.(3.7) and (3.10) we translate this value to observational transition values:

$$a_{III}(\eta_0) = \bar{A}_{II}\left(\frac{R}{M}\right)^R {t_0}^M \, {t_m}^{R-M} \tag{3.11}$$

The last point is to make an approach for the dilaton field. This is considered practically constant from the beginning of radiation dominated stage until the current time. Their value can be supposed coincident with the value at exit inflation time in a sudden but not continuous transition for the dilaton, since no one of their temporal derivatives can match this asymptotic behaviours.

$$\phi_{II} \;=\; \phi(\eta_1) \;=\; \phi_1 \tag{3.12}$$

Remember the expression for dilaton in inflation dominated stage, that gives

$$\phi_{II} \;=\; \phi_I + 2d \ln a(\eta_1) \tag{3.13}$$

temporal dependence parameters		*String Driven Cosm.B.* d	$d=3$
Cosmic Time description	infl. **Q**	$\frac{2}{d+1}$	$\frac{1}{2}$
	rad. **R**	$\frac{2}{d+1}$	$\frac{1}{2}$
	mat. **M**	$\frac{2}{d}$	$\frac{2}{3}$
Conformal Time description	infl. **q**	$\frac{2}{d+3}$	$\frac{1}{3}$
	rad. **r**	$\frac{2}{d-1}$	1
	mat. **m**	$\frac{2}{d-2}$	2

Table 2: Temporal Dependences in the String Driven Cosmological Backgrounds

Values of the exponents in the temporal dependences for the inflationary, radiation dominated and matter dominated stages in String Driven Cosmological Background. The parameters Q, R and M appear in the cosmic time description, whereas the parameters q, r and m are in the conformal time one. Particularization to the three dimensional case is given.

4 Generation of Gravitational Wave Perturbations in String Cosmology

After obtaining the minimal cosmological model for the scale factor in the framework of string cosmology, we will compute one of its important observational consequences, the stochastic background of relic gravitational waves. As first noticed by Grishchuk [6], relic gravitational waves are inevitably generated in practically all cosmological models oughted to their variable gravitational field.

The generation of gravitational wave backgrounds can be studied classically as propagation and amplification by the expanding background of one initial spectrum of tensorial perturbations on the metric. From the quantum mechanical point of view, a particle production mechanism occurs and gravitational waves of every wavelenght are generated as gravitons from an initial vacuum state. This translates into a combined classical-quantum mechanical formalism. (See for instance [7] and [26]). Gravitons produced at scales near the Planck time must exist today in the form of gravitational waves of wavelenghts to be determined by the propagation equation. [1].

The number or particles produced between an initial -asymptotic- vacuum state and a final state, where particles -gravitons- has been created by the effect of the changing gravitational field can be obtained by computing the Bogoliubov coefficients [4]. Grishchuk [6] has a similar approach modelizing the changing gravitational field as a potential barrier. Each polarization degree of freedom of tensorial perturbation satisfies the same propagation equation that a massless scalar field ([3],[5],[9]). But in frameworks with a dilaton field, attention must be put in accounting for the dilaton contribution to the gravitational field change and to the metric perturbation, breaking the validity of reported treatments [15].

In the next, we summarize and follow the treatment of tensorial perturbations in cosmological backgrounds within the Brans-Dicke frame developped in ref.[15]. That is necessary for accounting of the full effect of the dilaton field and for the consistency with the framework in which the string cosmology solutions have been obtained. We will show that this formalism includes as a particular case (no dilaton field) the formalism developped by Allen [4] and Grishchuk [6] in the framework of quantum field theory.

The general action written in Brans-Dicke frame takes the form:

$$S = -\frac{c^3}{16\pi G_D} \int d^{d+1}x \sqrt{\mid g \mid} e^{-\phi} \left(R - \omega \, g^{\mu\nu} \, \partial_\mu \phi \, \partial_\nu \phi \right) + S_M \qquad (4.1)$$

where S_M represents the contribution of matter sources such that its stress tensor is $\sqrt{\mid g \mid} T_{\mu\nu} = 2 \frac{\delta S_M}{\delta g^{\mu\nu}}$, antisymmetric tensors and dilaton potential are vanishing and

ω is the usual Brans-Dicke parameter. For $\omega = \infty$ General Relativity expressions are found and for $\omega = -1$ this action coincides with the low energy effective string action eq.(2.2). By varying with respect to the metric $g_{\mu\nu}$ and to dilaton ϕ the following two equations are obtained:

$$R_\mu{}^\nu + \nabla_\mu \nabla^\nu \phi + (\omega + 1)[\delta_\mu{}^\nu((\nabla\phi)^2 - \Box\phi) - \nabla_\mu\phi\nabla^\nu\phi] = \frac{8\pi G_D}{c^4} e^\phi T_\mu{}^\nu \tag{4.2}$$

$$R + \omega(\nabla\phi)^2 - 2\omega\,\Box\phi = 0 \tag{4.3}$$

where $\Box = \xi^{\mu\nu}\nabla_\mu\nabla_\nu$ is the covariant D'Alambertian operator. Again, the case $\omega = -1$ recoveres the low energy string effective equations (2.4).

The free-linearized wave equation for a generic metric fluctuation $h_{\mu\nu} = \delta g_{\mu\nu}$ is obtained by expressing the perturbed metric as $\hat{g}_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$, and all the sources are considered fixed $\delta T_\mu{}^\nu = \delta\phi = 0$, second order (and higher) terms in $h_{\mu\nu}$ are neglected.

In the following, the study is made only for the tensorial component of the metric fluctuation $h_{\mu\nu}$. The general case for the metric is considered: an anisotropic background, with d expanding external spatial dimensions with scale factor $a(t)$ and n contracting internal dimensions with scale factor $b(t)$, both flat and Euclidean. Expressed in the synchronous gauge, the metric is:

$$g_{\mu\nu} = (1, -a^2(t)\gamma_{ij}(x), -b^2(t)\gamma_{ab}(y)) \tag{4.4}$$

where again the component (0,0) is written in lenght units and corresponds to the variable $t = c\mathcal{T}$, indexes i, j run from $(1 \ldots d)$ and indexes a, b run from $(d+1 \ldots d+n+1)$.

The metric perturbation is considered propagating only in the external spatial dimensions, decoupled from the sources and purely tensorial:

$$h_{\mu\nu} = h_{\mu\nu}(\mathbf{x}, t) \tag{4.5}$$

$$h_{0\mu} = h_{a\mu} \quad = \quad 0 \tag{4.6}$$

That means, a pure gravitational wave that can be expressed in the transverse-traceless gauge, thus $g^{\mu\nu}\ h_{\mu\nu} = 0$ and $\nabla_\nu h_\mu{}^\nu = 0$. With these conditions, eq.(4.3) is trivially satisfied and from (4.2) the free-linearized wave equation for the tensorial metric perturbation $h_{\mu\nu}$ is obtained:

$$\delta R_\mu{}^\nu + \frac{1}{2}\,\dot{\phi}\,\dot{h}_{\mu\alpha}\,g^{\nu\alpha} - h^{\nu\alpha}\nabla_\mu\nabla_\alpha\phi = 0 \tag{4.7}$$

It must be noticed that this equation is ω-independent and therefore it is valid in general for all Brans-Dicke cases and in particular for the low energy effective string action. Eq.(4.7) takes a shorter form when the whole information about the metric is introduced [15].

$$\Box h_i{}^j - \dot{\phi}\dot{h}_i{}^j = 0 \tag{4.8}$$

Writting the perturbation in terms of eigenstates of the Laplace operator of wavenumber k: $\nabla^2 h_i^{\,j}(k) = -k^2 h_i^{\,j}(k)$, eq. (4.8) takes the form:

$$\ddot{h_i^{\,j}} + (dH + nF - \dot{\phi})\dot{h_i^{\,j}} + \left(\frac{k}{a}\right)^2 h_i^{\,j} = 0 \tag{4.9}$$

here dot means derivative with respect to t (remember $t = c\mathcal{T}$), $H = \frac{\dot{a}}{a}$ and $F = \frac{\dot{b}}{b}$. The most suitable form of eq.(4.9) is obtained by writting it in conformal time $d\eta = \frac{dt}{a(t)}$. It is convenient to define a variable ψ:

$$\psi_i^{\,j} = h_i^{\,j}\; a^{\frac{d-1}{2}}\; b^{\frac{n}{2}}\; e^{-\frac{\phi}{2}} \quad , \tag{4.10}$$

in such a way that $\psi_i^{\,j}$ is identified with each one of the two polarization modes of the perturbation h_{ij}. From eq.(4.9) it is immediately seen that each polarization mode ψ satisfies the corresponding equation written in conformal time:

$$\psi^{''} + (k^2 - V(\eta))\psi = 0 \tag{4.11}$$

where :

$$\begin{aligned} V(\eta) = {} & \frac{d-1}{2}\frac{a^{''}}{a} + \frac{n}{2}\frac{b^{''}}{b} - \frac{\phi^{''}}{2} + \frac{1}{4}(d-1)(d-3)\left(\frac{a^{'}}{a}\right)^2 + \frac{1}{4}n(n-2)\left(\frac{b^{'}}{b}\right)^2 \\ & + \frac{1}{4}\phi^{'2} + \frac{1}{2}n(d-1)\frac{a^{'}b^{'}}{a\,b} - \frac{1}{2}(d-1)\left(\frac{a^{'}}{a}\right)\phi^{'} - \frac{n}{2}\left(\frac{b^{'}}{b}\right)\phi^{'} \end{aligned} \tag{4.12}$$

Thus, the problem of metric fluctuation propagation is reduced to a second order differential Schrödinger type equation, where the "potential" $V(\eta)$ takes into account the conformal time derivatives of the background as well as those of dilaton fields; that means, the whole variable gravitational field "pumping" the generation and amplification of gravitons.[15]

Following our definition of the variable $t = c\mathcal{T}$ in length units, the conformal time variable η has also length units. Thus, the "potential" (4.12) have units $(\text{length})^{(-2)}$ and the wavenumber k has units $(\text{length})^{(-1)}$.

As it was pointed out in [15], this treatment generalizes to higher dimensional backgrounds the treatment more widely work, usually based on the pioneering works of Grishchuck [1], [5]. It generalizes also other early works like those of Allen [4].

4.1 Particular Cases

String Driven Case

In order to compute perturbations in our cosmological background, we will consider the suitable particularization of eqs.(4.11) and (4.12). For arbitrary d and $n = 0$ eqs.(4.11) and (4.12) take the form:

$$\psi^{''} + (k^2 - V(\eta))\psi = 0 \tag{4.13}$$

$$\begin{aligned} V(\eta) = & \frac{d-1}{2}\frac{a^{''}}{a} - \frac{\phi^{''}}{2} + \frac{1}{4}(d-1)(d-3)\left(\frac{a^{'}}{a}\right)^2 \\ & + \frac{1}{4}\phi^{'2} - \frac{1}{2}(d-1)\left(\frac{a^{'}}{a}\right)\phi^{'} \end{aligned} \tag{4.14}$$

The perturbation variable ψ is written as:

$$\psi_i^{\ j} = h_i^{\ j} a^{\frac{d-1}{2}} e^{-\frac{\phi}{2}} \quad , \tag{4.15}$$

and as consequence, $\psi_0^\mu = \psi_\mu^0 = 0$. Equivalently, the tensorial perturbation is:

$$\begin{aligned} h_{ij} &= \psi_i^{\ j} a^{-\frac{d+3}{2}} e^{\frac{\phi}{2}} \\ h_{0\,\mu} = h_{\mu\,0} &= 0 \\ h_{a\,\mu} = h_{\mu\,a} &= 0 \end{aligned} \tag{4.16}$$

where the index μ runs now from $1 \ldots d$.

No Dilaton Case

It is possible to extract from eqs.(4.13), (4.14) and (4.15) the case without dilaton field, in which we obtain the set of equations:

$$\psi^{''} + (k^2 - V(\eta))\psi = 0 \tag{4.17}$$

$$V(\eta) = \frac{d-1}{2}\left(\frac{a^{''}}{a}\right) + \frac{1}{4}(d-1)(d-3)\left(\frac{a^{'}}{a}\right)^2 \tag{4.18}$$

The perturbation variable ψ is:

$$\psi_i^{\ j} = h_i^{\ j}\, a^{\frac{d-1}{2}} \tag{4.19}$$

and the metric tensorial perturbation is:

$$\begin{aligned} h_{i\,j} &= \psi_i^{\ j}\, a^{-\frac{d+3}{2}} \\ h_{0\,\mu} = h_{\mu\,0} &= 0 \end{aligned}$$

The case of three spatial dimensions gives the simplest expressions:

$$\psi'' + (k^2 - V(\eta))\psi = 0 \tag{4.20}$$

$$V(\eta) = \left(\frac{a''}{a}\right) \tag{4.21}$$

where the perturbation is simply written

$$\psi_i^{\ j} = a\ h_i^{\ j} \tag{4.22}$$

with $h_{i\,j} = \psi_i^{\ j}\ a^{-3}$ and $h_{0\,\mu} = h_{\mu\,0} = 0$. This particular (3+1)-dimensional case gives the same set of equations ((4.20)-(4.22)) that can be found in Grischuhck treatments [5]. These equations can be also identified with the formalism found in [4].

For the sake of completeness, we compare now with the linearized Einstein equations for an homogeneus, isotropic and spatially flat metric $g_{\mu\nu}$ which in the conformal time gauge is:

$$ds^2 = a(\eta)^2(-d\eta^2 + d\mathbf{x}^2) \tag{4.23}$$

A gravitational perturbation $h_{\mu\nu} = a^2(\eta)f_{\mu\nu}$ of comoving wave number $\mathbf{k}$ is overimposed such that:

$$\hat{g}_{\mu\nu} = a^2(\eta)(\eta_{\mu\nu} + f_{\mu\nu}) \tag{4.24}$$

By performing a Fourier expansion, $f_{\mu\nu}$ can be described as:

$$f_{\mu\nu}(\eta, \mathbf{x}) = \varphi(\eta)e_{\mu\nu}(\mathbf{k})\exp(\imath\mathbf{kx}) \tag{4.25}$$

where $e_{\mu\nu}$ is the constant polarization vector. It can be shown that the amplitude perturbation satisfies the equation [4]:

$$\varphi''(\eta) + (d-1)\frac{a'(\eta)}{a(\eta)}\varphi'(\eta) + k^2\varphi = 0 \tag{4.26}$$

where $k = \mid \mathbf{k} \mid$, $\mathbf{k}$ is the comoving wave number and prime stands for conformal temporal derivatives. This equation is equivalent to eq.(4.9) in the Brans-Dicke formalism, provided particularization to isotropic cases ($F = 0$) and no dilaton dynamics ($\dot{\phi} = 0$). For $d = 3$, eq.(4.26) is the known equation found in the literature ([4]).

Let us write:

$$\varphi(\eta) = a(\eta)^{-\left(\frac{d-1}{2}\right)}v(\eta) \quad .$$

Then, we have

$$u''(\eta) + \left[\frac{d-1}{2}\left(\frac{3-d}{2}\left(\frac{a'(\eta)}{a(\eta)}\right)^2 - \frac{a''(\eta)}{a(\eta)}\right) + k^2\right]u(\eta) = 0 \tag{4.27}$$

that, in the particular three spatial dimensions take the form:

$$\varphi(\eta) = \frac{u(\eta)}{a(\eta)}$$
$$v'(\eta) = -\frac{a'(\eta)}{a(\eta)} v(\eta)$$
$$u''(\eta) + (k^2 - V(\eta))u(\eta) = 0 \qquad V(\eta) = \frac{a''(\eta)}{a(\eta)}$$

This shows the equivalence with the Grishchuk formalism and with the (3+1)-dimensional particularization of the No Dilaton Case showed above.

4.2 The Power Spectrum

We perform an expansion of the vacuum state (amplitude of perturbations at the classical level) in positive- and negative- frequency modes in each stage. The Bogoliubov coefficients mix the modes before and after the transition or the graviton creation-annihilation operators a_k and ${a_k}^+$. $\mid \beta_k \mid^2$ gives the number of particles of comoving frequency k generated on the vacuum state. This is the "initial" state (at a time sufficiently early before the transition to radiation dominated stage), so the corresponding Bogoliubov coefficient in the radiation dominated stage yields the total number of particles created during the whole inflationary era with respect that initial state.

We will determine the Bogoliubov coefficients $\alpha(k)$ and $\beta(k)$ for every wavenumber perturbation k by considering a continous matching of the perturbation amplitude at the transition time η_1 between inflation and radiation dominated stages. Thus, we will impose continuity of the amplitude perturbation -treated as quantum state- as well as of its first conformal time derivative at the sudden transition at conformal time η_1. This is similar to work with the picture of a potential barrier generated by the variable gravitational field [5]. Gravitational waves would interact with this barrier. By describing one state "out" the barrier, one state "in" with frozen amplitude and a comeback to "out", the Bogoliubov coefficients can be computed. But as above said, the contribution of the dilaton change can be missed.

The stochastic background of gravitational waves generated in the cosmological transitions is described by the power spectrum $P(\omega)d\omega$, which is the energy density contained in gravitational waves with frequencies in the interval between ω and $\omega + d\omega$; ω is the proper frequency at the detection time related to the comoving wavenumber k through:

$$\omega = \frac{c\, k}{a(\eta_0)} \tag{4.28}$$

The total energy density in gravitational waves is computed as $\rho_{GW} = \int P(\omega)d\omega$ [4]. The power spectrum $P(\omega)d\omega$ is given by $P(\omega)d\omega = 2\hbar\omega |\ \beta\ |^2 d\omega dN$ where dN is the density of states, which in three spatial dimensions (as we have at the detection moment) is $dN = \frac{\omega^2}{2\pi^2 c^3} d\omega$; $|\ \beta\ |^2$ is the number of gravitons created in each frequency interval and each polarization state. This comes multiplied by the two polarization degrees of freedom of the tensorial metric perturbation and by the energy $\hbar\omega$ of each one:

$$P(\omega)d\omega = \frac{\hbar\omega^3}{\pi^2 c^3} |\ \beta\ |^2 d\omega \tag{4.29}$$

Thus, the power spectrum $P(\omega)$ has dimensions $[P(\omega)] = \frac{\text{energy time}}{\text{volume}} = \frac{\text{mass}}{\text{time lenght}}$.

It is usual to write the power spectrum as fraction of the critical energy density by octave, which is the closest form to gravitational wave detection procedures:

$$\Omega_{GW} = \frac{1}{\rho_c} \frac{d\rho_{GW}(\omega)}{d\ln\omega}$$

here ρ_{GW} is the energy density contained in gravitational waves and ρ_c is the critical energy density $\rho_c = \frac{3\,H_0{}^2}{8\pi G}$. This is equivalent to compute

$$\Omega_{GW} = \frac{\omega}{\rho_c} P(\omega) \tag{4.30}$$

Obviously, Ω_{GW} is dimensionless $[\Omega_{GW}] = 1$.

5 Power Spectrum in String Driven Cosmology

5.1 No Dilaton Case

We begin our computations of power spectrum of gravitational waves with the simplest case: we will neglect in this the whole dilaton role. Thus, we will solve eqs.(4.17), (4.18) and the reduced amplitud perturbation given by (4.19).

In the inflationary stage, from eqs.(3.5) and (4.18), the "potential" $V(\eta)$ is given by:

$$V(\eta) = \left(\left(\frac{d-1}{2} q + \frac{1}{2} \right)^2 - \frac{1}{4} \right) (\eta_I - \bar{\bar{\eta}})^{-2} \tag{5.1}$$

The general solution of eq.(4.17) with the potential (5.1) is given in terms of Bessel functions:

$$\psi = (\eta_I - \bar{\bar{\eta}})^{\frac{1}{2}} \left(C_1 \mathcal{H}_\nu{}^{(1)}(k(\eta_I - \bar{\bar{\eta}})) + C_2 \mathcal{H}_\nu{}^{(2)}(k(\eta_I - \bar{\bar{\eta}})) \right) \tag{5.2}$$

with index ν:

$$\nu = \pm\frac{1}{2}((d-1)q+1) \tag{5.3}$$

Or, from eq.(2.7) in terms of the number d of the spatial dimensions.

$$\nu = \frac{3d+1}{2(d+3)} \tag{5.4}$$

It gives $\nu = \frac{5}{6}$ for three spatial dimensions.

The coefficients C_1 and C_2 are fixed by the choice of initial or boundary conditions, i.e. by the inflationary vacuum state [4]. Thus, the asymptotic expression of (5.2) for early times should contain only positive frequency modes.

The argument of the Hankel functions involved can be expressed in terms of the cosmic time. From eq.(3.6):

$$z \equiv k(\eta_I - \bar{\bar{\eta}}) = k\left(\eta_1(1+\frac{q}{r}) - \bar{\bar{\eta}}\right) \quad , \tag{5.5}$$

and with the expressions relating the conformal time description with the observational transition times and cosmic time eqs.(3.7), (3.8) and (3.9), we get:

$$z = k\left(\frac{R}{M}t_r\right)^{1-R} \frac{Q}{R(Q+1)}\frac{1}{\bar{A}_{II}}\left[\left(\frac{t_I - t}{t_I - t_r}\right)^{Q+1}\right] \tag{5.6}$$

As discussed in section II (see eq.(2.6) and comments below), the parameter t_I is always greater than the exit of inflationary stage t_r, thus the argument z is real and $z > 0$ for all $t < t_r$. For time very early in the inflationary stage $t \rightarrow t_i$ and $t << t_r$, the argument z approaches its maximum value. Thus, we must identify the large argument regime with the vacuum state at enough early inflationary stage (usually refered as well below the transition to radiation dominated epoch).

The asymptotic behaviours of Hankel functions [27]: and the positive frequency mode condition at early stages, fix $C_2 = 0$. The remaining freedom in the coefficient C_1 is linked to the normalization of the vacuum inflationary state and we include it in the normalization coefficient $\mathcal{N}$. In this way, the reduced amplitude perturbation in the inflationary stage (before the transition) is written as:

$$\psi_I = \mathcal{N}(\eta_I - \bar{\bar{\eta}})^{\frac{1}{2}}\mathcal{H}_\nu{}^{(1)}\left(k(\eta_I - \bar{\bar{\eta}})\right) \tag{5.7}$$

In the radiation dominated stage, from the scale factor eq.(3.5), the potential (4.18) takes the form:

$$V(\eta) = \left(\left(\frac{d-1}{2}r - \frac{1}{2}\right)^2 - \frac{1}{4}\right)\bar{\eta}^{-2} \tag{5.8}$$

And the general solution of eq.(4.17), with eq.(5.8) is given again in terms of Hankel functions:

$$\psi = \bar{\eta}^{\frac{1}{2}} \left(\mathcal{D}_1 \mathcal{H}_\mu{}^{(1)}(k\bar{\eta}) + \mathcal{D}_2 \mathcal{H}_\mu{}^{(2)}(k\bar{\eta}) \right) \quad , \tag{5.9}$$

but now with index μ:

$$\mu = \pm \frac{1}{2}((d-1)r - 1) \quad . \tag{5.10}$$

Since the radiation dominated stage has $r = \frac{2}{d-1}$ (eq.(2.9)), it makes:

$$\mu = \pm \, \frac{1}{2} \tag{5.11}$$

In this case, by using the properties of Hankel functions with fractional index (See [27]) , ψ eq.(5.9) has the expression:

$$\psi = -i \sqrt{\frac{2}{\pi k}} \left(\mathcal{D}_1 e^{ik\bar{\eta}} - \mathcal{D}_2 e^{-ik\bar{\eta}} \right) \tag{5.12}$$

which can be written also in the way

$$\psi_{II} = i \, \mathcal{M} \sqrt{\frac{2}{\pi k}} \left(\alpha e^{-ik(\bar{\eta}-\eta_1)} + \beta e^{ik(\bar{\eta}-\eta_1)} \right) \quad . \tag{5.13}$$

Here we have defined $\mathcal{M}$ a suitable normalization constant and the coefficients $\alpha = \frac{\mathcal{D}_2}{\mathcal{M}} \, e^{-\, ik\eta_1}$ and $\beta = -\frac{\mathcal{D}_1}{\mathcal{M}} \, e^{ik\eta_1}$ are the Bogoliubov's coefficients, since they are multiplying the positive- and negative-frequency pure modes.

With the expressions for the reduced amplitude perturbation in each stage (5.7) and (5.13), where we remember the parameter ν has the value given by (5.3), we will determine the value of Bogoliubov's coefficients α and β by matching the reduced amplitude perturbation and its first conformal time derivative continuously across the inflation-radiation dominated transition value η_1.

$$\psi_I(\bar{\bar{\eta}} = \eta_1) = \psi_{II}(\bar{\eta} = \eta_1) \tag{5.14}$$

$$\left. \frac{d\psi_I}{d\bar{\bar{\eta}}} \right|_{\bar{\bar{\eta}} \, = \, \eta_1} = \left. \frac{d\psi_{II}}{d\bar{\eta}} \right|_{\bar{\eta} \, = \, \eta_1} \tag{5.15}$$

The derivative of the reduced amplitude perturbation can be computed as:

$$\begin{aligned} \frac{d\psi_I}{d\bar{\bar{\eta}}} &= -\frac{1}{2} \mathcal{N} \, (\eta_I - \bar{\bar{\eta}})^{-\frac{1}{2}} \, \mathcal{H}_\nu{}^{(1)} \, (k(\eta_I - \bar{\bar{\eta}})) \; + \\ & \quad - \mathcal{N} k (\eta_I - \bar{\bar{\eta}})^{\frac{1}{2}} \frac{d\mathcal{H}_\nu{}^{(1)} \, (k(\eta_I - \bar{\bar{\eta}}))}{d \, (k(\eta_I - \bar{\bar{\eta}}))} \end{aligned} \tag{5.16}$$

$$\frac{d\psi_{II}}{d\bar{\eta}} = i\mathcal{M} \sqrt{\frac{2}{\pi k}} \left((-ik) \alpha e^{-ik(\bar{\eta}-\eta_1)} + (ik) \beta e^{ik(\bar{\eta}-\eta_1)} \right) \tag{5.17}$$

In the next, we will call:

$$\mathcal{H}'_{\nu}{}^{(i)} \;=\; \left.\frac{d\mathcal{H}_{\nu}{}^{(i)}\,(k(\eta_I - \bar{\bar{\eta}}))}{d((k(\eta_I - \bar{\bar{\eta}})))}\right|_{\bar{\bar{\eta}}=\eta_1} \tag{5.18}$$

In particular, the argument of the Hankel functions involved in the matching can be expressed, following eq. (3.6)

$$k(\eta_I - \bar{\bar{\eta}})|_{\bar{\bar{\eta}}\,=\,\eta_1} \;=\; k\;\eta_1\,\frac{q}{r} \;=\; X \tag{5.19}$$

With this property, equations (5.14) and (5.15) can be written as:

$$\mathcal{N}(X)^{\frac{1}{2}}\mathcal{H}_{\nu}{}^{(1)}\,(X) = i\mathcal{M}\sqrt{\frac{2}{\pi}}(\alpha+\beta) \tag{5.20}$$

$$\frac{1}{2k}\mathcal{N}(X)^{-\frac{1}{2}}\mathcal{H}_{\nu}{}^{(1)}\,(X) + k\mathcal{N}(X)^{\frac{1}{2}}\mathcal{H}'_{\nu}{}^{(1)}\,(X) = \mathcal{M}\sqrt{2\pi}(\beta-\alpha) \tag{5.21}$$

By solving this system, we find for the Bogoliubov's coefficients:

$$\alpha = -\sqrt{\frac{\pi}{2}}\,\frac{\mathcal{N}}{2\mathcal{M}}X^{\frac{1}{2}}\left[\left(\frac{1}{2X}\;+\;i\right)\mathcal{H}_{\nu}{}^{(1)}(\,X\,)\;+\;\mathcal{H}'_{\nu}{}^{(1)}(\,X\,)\right] \tag{5.22}$$

$$\beta = \sqrt{\frac{\pi}{2}}\,\frac{\mathcal{N}}{2\mathcal{M}}X^{\frac{1}{2}}\left[\left(\frac{1}{2X}\;-\;i\right)\mathcal{H}_{\nu}{}^{(1)}(\,X\,)\;+\;\mathcal{H}'_{\nu}{}^{(1)}(\,X\,)\right] \tag{5.23}$$

From the properties of the Hankel functions (See [28]), and their conjugated, for real ν and X which is the case here:

$$\overline{\mathcal{H}_{\nu}{}^{(1)}(\,X\,)} = \mathcal{H}_{\nu}{}^{(2)}(\,X\,) \tag{5.24}$$

$$\overline{\mathcal{H}'_{\nu}{}^{(1)}(\,X\,)} = \mathcal{H}'_{\nu}{}^{(2)}(\,X\,) \tag{5.25}$$

The conjugate of Bogoliubov's coefficients $\bar{\alpha}$ and $\bar{\beta}$ can be written as:

$$\bar{\alpha} = -\sqrt{\frac{\pi}{2}}\,\frac{\bar{\mathcal{N}}}{2\bar{\mathcal{M}}}X^{\frac{1}{2}}\left[\left(\frac{1}{2X}-i\right)\mathcal{H}_{\nu}{}^{(2)}(\,X\,)\;+\;\mathcal{H}'_{\nu}{}^{(2)}(\,X\,)\right] \tag{5.26}$$

$$\bar{\beta} = \sqrt{\frac{\pi}{2}}\,\frac{\bar{\mathcal{N}}}{2\bar{\mathcal{M}}}X^{\frac{1}{2}}\left[\left(\frac{1}{2X}+i\right)\mathcal{H}_{\nu}{}^{(2)}(\,X\,)\;+\;\mathcal{H}'_{\nu}{}^{(2)}(\,X\,)\right] \tag{5.27}$$

With the properties of Hankel functions (See [28]) and their wronskian (See [27]):

$$\left\{\mathcal{H}_{\nu}{}^{(1)}(z),\;\mathcal{H}_{\nu}{}^{(2)}(z)\right\} \;=\; -i\,\frac{4}{\pi z} \tag{5.28}$$

and by rearranging coefficients, we obtain the expressions for the square modulii of Bogoliubov's coefficients.

$$\begin{aligned}
\mid \alpha \mid^2 = \frac{\pi}{8} \frac{\mid \mathcal{N} \mid^2}{\mid \mathcal{M} \mid^2} \frac{\left(\nu - \frac{1}{2}\right)^2}{X} & \left[\left(1 + \frac{X^2}{\left(\nu - \frac{1}{2}\right)^2} \right) \mathcal{H}_\nu{}^{(1)}(X) \mathcal{H}_\nu{}^{(2)}(X) + \right. \\
& + \frac{X^2}{\left(\nu - \frac{1}{2}\right)^2} \mathcal{H}_{\nu-1}{}^{(1)}(X) \mathcal{H}_{\nu-1}{}^{(2)}(X) + \\
& \left. - \frac{2X}{\left(\nu - \frac{1}{2}\right)} \mathcal{H}_\nu{}^{(1)}(X) \mathcal{H}_{\nu-1}{}^{(2)}(X) - i \frac{4}{\pi \left(\nu - \frac{1}{2}\right)} + \frac{4\, X}{\pi \left(\nu - \frac{1}{2}\right)^2} \right] \qquad (5.29)
\end{aligned}$$

$$\begin{aligned}
\mid \beta \mid^2 = \frac{\pi}{8} \frac{\mid \mathcal{N} \mid^2}{\mid \mathcal{M} \mid^2} \frac{\left(\nu - \frac{1}{2}\right)^2}{X} & \left[\left(1 + \frac{X^2}{\left(\nu - \frac{1}{2}\right)^2} \right) \mathcal{H}_\nu{}^{(1)}(X) \mathcal{H}_\nu{}^{(2)}(X) + \right. \\
& + \frac{X^2}{\left(\nu - \frac{1}{2}\right)^2} \mathcal{H}_{\nu-1}{}^{(1)}(X) \mathcal{H}_{\nu-1}{}^{(2)}(X) + \\
& \left. - \frac{2X}{\left(\nu - \frac{1}{2}\right)} \mathcal{H}_\nu{}^{(1)}(X) \mathcal{H}_{\nu-1}{}^{(2)}(X) - i \frac{4}{\pi \left(\nu - \frac{1}{2}\right)} - \frac{4\, X}{\pi \left(\nu - \frac{1}{2}\right)^2} \right] \qquad (5.30)
\end{aligned}$$

By requiring the Bogoliubov's coefficients satisfy the usual normalization condition:

$$\mid \alpha \mid^2 - \mid \beta \mid^2 = 1 \qquad (5.31)$$

we have the constraint:

$$\frac{\mid \mathcal{N} \mid^2}{\mid \mathcal{M} \mid^2} = 1 \qquad (5.32)$$

This condition holds for all k, thus it ensure us the absence of isolated dependence of k through the normalization constants. The difference among them is only a phase. The same result can be reached by computing explicitly the constants $\mathcal{N}$ and $\mathcal{M}$ in the framework of a specific normalization scheme.

5.1.1 The Power Spectrum in Physical Units

Now we are able to compute the power spectrum of the stochastic gravitational wave background with full physical meaning. In terms of the proper frequency ω at detection time eq.(4.28), we read the variable X eq.(5.19) as:

$$X = \omega \left(\frac{q}{r}\right) \frac{a(\eta_0)\eta_1}{c} \qquad (5.33)$$

Notice that the dependence with the global scale factor parameter $\bar{A_{II}}$ (the only quantity unknown in these expressions), is cancelled in X by the product of $\eta_1 a(\eta_0)$. With the help of eqs.(3.8), (3.11), (3.7) and some rearrangements we have:

$$X \equiv \omega\, S \quad , \quad S \;=\; \frac{Q}{(Q+1)M}\left(\frac{\mathcal{T}_0}{\mathcal{T}_m}\right)^M \left(\frac{\mathcal{T}_m}{\mathcal{T}_r}\right)^R \mathcal{T}_r \tag{5.34}$$

That is, we find the argument X related to the physical proper frequency in a way *totally determined* by the cosmological model studied, and linked to observational times. The ratii $\left(\frac{\mathcal{T}_0}{\mathcal{T}_m}\right)^M$ and $\left(\frac{\mathcal{T}_m}{\mathcal{T}_r}\right)^R$ are exactly the scale factor expansion reached during the radiation dominated and matter dominated epochs, respectively, introduced as part of our linking to observational Universe information on the minimal cosmological model. (See section III). Both values are strongly constrained by the observational evolution data. Meanwhile, the value $\mathcal{T}_r$ (beginning of radiation dominated stage) gives us the order of magnitude of the exit of inflationary stage, related to the order of magnitude of GUT scales.

Obviously, the quantity S has dimensions $[S] =$ time. Thus, as $[\omega] =$ rad (time)$^{-1}$, X is dimensionless. We can give here the value of S for our cosmological scale factor. From String Theory, we have obtained the scale factors with parameters of temporal dependence $Q = \frac{2}{d+1}$, $R = \frac{2}{d+1}$ and $M = \frac{2}{d}$ for inflationary, radiation dominated and matter dominated stages, respectively (See eqs.(2.6), (2.8), (2.16) and table(2)). By linking with the cosmological observational description, we have considered the standard values for $\mathcal{T}_r$, $\mathcal{T}_m$ and $\mathcal{T}_0$ given in previous section. This leads to the expression

$$X_{String} \;=\; \omega\, \frac{d}{(3+d)} \left(\frac{\mathcal{T}_0}{\mathcal{T}_m}\right)^{\frac{2}{d}} \left(\frac{\mathcal{T}_m}{\mathcal{T}_r}\right)^{\frac{2}{d+1}} \mathcal{T}_r \qquad .$$

For three spatial dimensions, and the time constants given in seconds, we have:

$$X_{String} \;=\; \omega\, \frac{1}{2} 10^{-\frac{20}{3}} \mathrm{seg} \tag{5.35}$$

If we define a characteristic frequency ω_x which makes unity the argument X ($\omega_x\, S \;=\; 1$), the value of it would be

$$\omega_x \;\sim\; 9.28\; 10^6\, \frac{\mathrm{rad}}{\mathrm{s}} \;\sim\; 1.48 \;\mathrm{Mhz} \tag{5.36}$$

Now, we compute the full and exact expression of the power spectrum for the stochastic background of gravitational waves produced in the inflation-radiation dominated transition. Following the usual definition, the power spectrum of gravitational waves computed in three spatial dimensions is:

$$P(\omega) d\omega = \frac{\hbar}{\pi^2 c^3} \omega^3 d\omega \mid \beta \mid^2 \tag{5.37}$$

From eq.(5.30) and eq.(5.34), we have:

$$P(\omega)d\omega = \frac{\hbar}{8\pi c^3}\omega^2\frac{\left(\nu-\frac{1}{2}\right)^2}{S}d\omega\left[\left(1+\frac{(\omega S)^2}{\left(\nu-\frac{1}{2}\right)^2}\right)\mathcal{H}_\nu{}^{(1)}(\omega S)\mathcal{H}_\nu{}^{(2)}(\omega S)+\right.$$
$$+\frac{(\omega S)^2}{\left(\nu-\frac{1}{2}\right)^2}\mathcal{H}_{\nu-1}{}^{(1)}(\omega S)\mathcal{H}_{\nu-1}{}^{(2)}(\omega S)\ +$$
$$\left.-\frac{2\,\omega S}{\left(\nu-\frac{1}{2}\right)}\mathcal{H}_\nu{}^{(1)}(\omega S)\mathcal{H}_{\nu-1}{}^{(2)}(\omega S)-\frac{4\,\omega S}{\pi\left(\nu-\frac{1}{2}\right)^2}-i\frac{4}{\pi\left(\nu-\frac{1}{2}\right)}\right] \tag{5.38}$$

where S is given by eq.(5.34).

The expression in square brackets is dimensionless. All physical units are found in the term multiplying it and proportional to $\omega^2 d\omega$. For our scale factor in three spatial dimensions, the coefficient in front will take the value:

$$\frac{\hbar}{8\pi c^3}\,\frac{\left(\nu-\frac{1}{2}\right)^2}{S}\ \sim\ 1.603\ 10^{-54}\ \frac{\text{erg seg}^3}{\text{cm}^3} \tag{5.39}$$

thus the power spectrum $P(\omega)d\omega$ has dimensions of energy density.

Notice the dependence of $P(\omega)$ on the scale factor parameters through the parameter ν, eq.(5.3), characterizing the expression of the power spectrum and its ω dependence. This is the way the cosmological theory considered is imprinted on the gravitational wave spectrum. The parameter X eq.(5.34), is uniquely fixed by the scale factor and according with observational values. We will see in the next sections as the different dilaton roles and the different theories, although involving the same scale factor, led to different power spectra shapes.

It is also possible to express the power spectrum in terms of the Bessel functions $\mathcal{J}_\nu(z)$ and the Neumann functions $\mathcal{Y}_\nu(z)$ [28]:

$$\mathcal{H}_\nu{}^{(1)}(z) = \mathcal{J}_\nu(z) + i\mathcal{Y}_\nu(z) \tag{5.40}$$
$$\mathcal{H}_\nu{}^{(2)}(z) = \mathcal{J}_\nu(z) - i\mathcal{Y}_\nu(z) \tag{5.41}$$

with the Wronskian[27]

$$\{\mathcal{J}_\nu(z),\ \mathcal{Y}_\nu(z)\}\ =\ \frac{2}{\pi z}\quad . \tag{5.42}$$

The power spectrum is expressed as:

$$
\begin{aligned}
P(\omega)d\omega = & \frac{\hbar}{8\pi c^3}\omega^2\frac{\left(\nu-\frac{1}{2}\right)^2}{S}d\omega \\
& \left[\left(1+\frac{(\omega S)^2}{\left(\nu-\frac{1}{2}\right)^2}\right)\Big(\mathcal{J}_\nu(\omega S)\mathcal{J}_\nu(\omega S)+\mathcal{Y}_\nu(\omega S)\mathcal{Y}_\nu(\omega S)\Big)+\right. \\
& +\frac{(\omega S)^2}{\left(\nu-\frac{1}{2}\right)^2}\Big(\mathcal{J}_{\nu-1}(\omega S)\mathcal{J}_{\nu-1}(\omega S)+\mathcal{Y}_{\nu-1}(\omega S)\mathcal{Y}_{\nu-1}(\omega S)\Big)+ \\
& -\frac{2\,\omega S}{\left(\nu-\frac{1}{2}\right)}\Big(\mathcal{J}_\nu(\omega S)\mathcal{J}_{\nu-1}(\omega S)+\mathcal{Y}_\nu(\omega S)\mathcal{Y}_{\nu-1}(\omega S)\Big)+ \\
& \left.-\frac{4\,\omega S}{\pi\left(\nu-\frac{1}{2}\right)^2}\right]
\end{aligned}
\tag{5.43}
$$

Another expression for the power spectrum is in terms of only Bessel functions $\mathcal{J}_\nu$, $\mathcal{J}_{-\nu}$. In order to make this, we use the relations ([28], [27]):

$$
\begin{aligned}
\mathcal{H}_\nu{}^{(1)}(z) &= \frac{i}{\sin(\pi\nu)}\Big(\mathcal{J}_\nu(z)e^{-i\pi\nu}-\mathcal{J}_{-\nu}(z)\Big) \\
\mathcal{H}_\nu{}^{(2)}(z) &= \frac{i}{\sin(\pi\nu)}\Big(\mathcal{J}_{-\nu}(z)-e^{i\pi\nu}\mathcal{J}_\nu(z)\Big) \\
\{\mathcal{J}_\nu(z),\ \mathcal{J}_{-\nu}(z)\} &= -\frac{2\,\sin(\pi\nu)}{\pi z}
\end{aligned}
$$

We find the next expression dealing only Bessel functions:

$$
\begin{aligned}
P(\omega)d\omega = & \frac{\hbar}{8\pi c^3}\omega^2\frac{\left(\nu-\frac{1}{2}\right)^2}{S\ \sin^2(\pi\nu)}d\omega \\
& \left[\left(1+\frac{(\omega S)^2}{\left(\nu-\frac{1}{2}\right)^2}\right)\Big(\mathcal{J}_\nu(\omega S)\mathcal{J}_\nu(\omega S)+\mathcal{J}_{-\nu}(\omega S)\mathcal{J}_{-\nu}(\omega S)+\right. \\
& -2\cos(\pi\nu)\mathcal{J}_\nu(\omega S)\mathcal{J}_{-\nu}(\omega S)\Big)+ \\
& +\frac{(\omega S)^2}{\left(\nu-\frac{1}{2}\right)^2}\Big(\mathcal{J}_{\nu-1}(\omega S)\mathcal{J}_{\nu-1}(\omega S)+\mathcal{J}_{1-\nu}(\omega S)\mathcal{J}_{1-\nu}(\omega S)+ \\
& +2\cos(\pi\nu)\mathcal{J}_{\nu-1}(\omega S)\mathcal{J}_{1-\nu}(\omega S)\Big)+ \\
& -\frac{2\,\omega S}{\left(\nu-\frac{1}{2}\right)}\Big(\mathcal{J}_\nu(\omega S)\mathcal{J}_{\nu-1}(\omega S)-\mathcal{J}_{-\nu}(\omega S)\mathcal{J}_{1-\nu}(\omega S)\Big)+
\end{aligned}
$$

$$
- \frac{2\,\omega S\ \cos(\pi\nu)}{\left(\nu-\frac{1}{2}\right)}\Big(\mathcal{J}_\nu(\omega S)\mathcal{J}_{1-\nu}(\omega S)-\mathcal{J}_{-\nu}(\omega S)\mathcal{J}_{\nu-1}(\omega S)\Big)+
$$

$$
- \frac{4\,\omega S\sin^2(\pi\,\nu)}{\pi\left(\nu-\frac{1}{2}\right)^2}\Bigg] \tag{5.44}
$$

Eqs.(5.44) and (5.43) manifest explicitely the real character of the mathematical expressions involved in the power spectrum. In fact, the imaginary terms belonging to Hankel functions cancel exactly among them and with the free imaginary term. This is more easily seen looking in the last expression, where the power spectrum is fully expressed in Bessel functions (real functions when the argument is real) (See [29]) From the numerical point of view, expressions (5.43) or (5.44) are more convenient, since standard algorithms are limited in compute Hankel functions to very high precision at large arguments and it leds to wrong results in computing the above expressions. Expressions (5.43) and (5.44) allow extend 1-2 orders of magnitude more the numerical computations of the exact spectrum. In any case, in the next chapter we will overcome this limitation with analytical expressions for the regime of small and large arguments.

Finally, we compute the fraction of critical energy density in gravitational waves by octave. Following the definition eq.(4.30) and using the expression (5.38), we have:

$$
\Omega_{GW} = \frac{\hbar\,G}{3{H_0}^2c^5}\,\frac{\left(\nu-\frac{1}{2}\right)^2}{S}\,\omega^3\left[\left(1+\frac{(\omega S)^2}{\left(\nu-\frac{1}{2}\right)^2}\right)\mathcal{H}_\nu{}^{(1)}(\omega S)\mathcal{H}_\nu{}^{(2)}(\omega S)+\right.
$$

$$
+\frac{(\omega S)^2}{\left(\nu-\frac{1}{2}\right)^2}\,\mathcal{H}_{\nu-1}{}^{(1)}(\omega S)\mathcal{H}_{\nu-1}{}^{(2)}(\omega S)\ +
$$

$$
\left. - \frac{2\,\omega S}{\left(\nu-\frac{1}{2}\right)}\mathcal{H}_\nu{}^{(1)}(\omega S)\mathcal{H}_{\nu-1}{}^{(2)}(\omega S)-\frac{4\,\omega S}{\pi\left(\nu-\frac{1}{2}\right)^2}-i\frac{4}{\pi\left(\nu-\frac{1}{2}\right)}\right] \tag{5.45}
$$

Again, we will compute the value of the coefficient multiplying ω^3 and the expression in square brackets:

$$
\frac{\hbar\,G}{3{H_0}^2c^5}\,\frac{\left(\nu-\frac{1}{2}\right)^2}{S}\ \sim\ 2.488\ 10^{-46}\ \text{seg}^{-3} \tag{5.46}
$$

5.2 Partial Dilaton Case

In this case, we will consider the role of the dilaton field in the equations for the reduced amplitude perturbation ψ. We will see that this procedure still means a partial consideration of the dilaton role on the gravitational wave spectra. By now, we will solve eqs.(4.13) and (4.14) with the reduced amplitude perturbation given by (4.15).

Now, in the inflationary stage, we take into account not only the scale factor but also the dilaton field. Following eqs.(2.6) and (3.5) we have for the dilaton in this stage:

$$\phi(\bar{\bar{\eta}}) = \phi_I + 2d \ln a(\bar{\bar{\eta}}) \tag{5.47}$$

Thus, the potential eq.(4.14) takes now the form:

$$V(\eta) \;=\; \left(\left(\frac{d+1}{2} q - \frac{1}{2} \right)^2 - \frac{1}{4} \right) (\eta_I - \bar{\bar{\eta}})^{-2} \tag{5.48}$$

Then, we have again an equation whose solution is expressed in terms of Bessel functions, but with the parameter ν having the value:

$$\nu \;=\; \pm \left(\frac{d+1}{2} q - \frac{1}{2} \right) \tag{5.49}$$

Or, expressed in terms of the spatial number dimensions,

$$\nu \;=\; \frac{d-1}{2(d+3)} \tag{5.50}$$

that is, $\nu \;=\; \frac{1}{6}$ for three spatial dimensions, and fixing the positive sign in the eq.(5.49).

Notice the difference with the parameter ν eq.(5.4) obtained in the No Dilaton Case. The difference among both cases is:

$$\nu\big|_{\text{part.dil.}} \;=\; \nu\big|_{\text{no dil.}} \;+\; (1-q)$$

It leds to a totally similar expression to eq.(5.7) for the solution of the reduced amplitude perturbation:

$$\psi_I \;=\; \mathcal{N} (\eta_I - \bar{\bar{\eta}})^{\frac{1}{2}} \mathcal{H}_\nu{}^{(1)} \left(k(\eta_I - \bar{\bar{\eta}}) \right) \tag{5.51}$$

where ν is given now by eq.(5.49) and $\mathcal{N}$ is a normalization constant.

For the radiation dominated stage, the scale factor is given by eq.(3.5) and the dilaton field, according to eqs.(2.8), remains "frozen" in its constant value at the exit inflation time:

$$\phi_{II} \;=\; \phi(\bar{\bar{\eta}})|_{\bar{\bar{\eta}}\,=\,\eta_1} \tag{5.52}$$

Thus, we recover exactly the same expressions for the potential eq.(5.8), the general solution eq.(5.9), the parameter $\mu = \pm\frac{1}{2}$ and consequently, the same expression in terms of Bogoliubov's coefficients α and β:

$$\psi_{II} = i\mathcal{M}\sqrt{\frac{2}{\pi k}}\Big(\alpha e^{-ik(\bar{\eta}-\eta_1)} + \beta e^{ik(\bar{\eta}-\eta_1)}\Big) \tag{5.53}$$

with $\mathcal{M}$ a normalization constant.

The Bogoliubov's coefficients computed now have exactly the same formal expressions yet obtained in the last section, eqs.(5.29) and (5.30), since the reduced amplitude perturbations involved in the matching at the transition at η_1 are formally the same functions (5.7) and (5.13) in both cases. Only the parameter ν is different, now given by eq.(5.49).

Similarly, all the relations in order to compute the power spectrum eq.(5.38), the fraction of critical energy density Ω_{GW} eq.(5.45) and the expression of the argument X eq.(5.34) as function of the proper frequency ω, hold in this case. Notice that the argument X does not depend on the dilaton role; this quantity comes fixed from the evolution of the scale factor. Thus, we have again $P(\omega)d\omega$ and Ω_{GW} as given by eqs.(5.38) and (5.45) but with the parameter ν given in this case by eq.(5.49).

In three spatial dimensions, we have the same values for the parameter S eq.(5.35) and for the characteristic frequency ω_x eq.(5.36). The numerical values of coefficients multiplying the square brackets expressions in (5.38) and (5.45) would be, in principle, differents since they depend of the parameter ν. But in our case, in three spatial dimensions, the value of $(\nu - \frac{1}{2})^2$ is the same for both cases, giving the same numerical value for these coefficients too.

It must be observed that the normalization constants $\mathcal{N}$ and $\mathcal{M}$ are constrainted by the normalization condition of Bogoliubov's coefficients to satisfy the same relation found in the case before: $\mathcal{N}$ and $\mathcal{M}$ differ only in a phase. (See eqs.(5.32)). But the case of particular and explicit normalization applicable to the no dilaton case is not generalizable to the dilaton case. Now, both constants will remain as undetermined parameters in the amplitude perturbations.

5.3 The Full Dilaton Role

In this section, we will take into account the full dilaton role. In the last section, we have considered it through the reduced amplitude perturbation ψ, solution of eq.(4.13). But we have left unconsidered the dilaton role in the total amplitude perturbation $h_{\mu\nu}$. In order to include this, we will match in this case the metric amplitude perturbation $h_{\mu\nu}$ instead of only the reduced amplitude perturbation ψ.

In the framework of General Relativity, as the treatment of Grishchuk [5] and Allen [4], both procedures are equivalents, since the metric perturbation and the reduced amplitude differ only through a power of the scale factor. In those cases, and with a description with suitable properties of continuity for the scale factor and its first conformal time derivative like those we are dealing with, the final result for the gravitational wave spectra would be the same, since there are no lack of information by matching only a part of the metric perturbation, the reduced amplitude. The remaining part would give us redundant information.

But in the case of theories with dilaton fields, Brans-Dicke frames and in particular, string cosmology, the remaining part in the total amplitude perturbation contains additional information with respect to the reduced one. It contains an added dependence on the dilaton field, as can be seen from eq.(4.16). The behaviour of this part through the transition is not the same as the scale factor, since there is no continuity in the first derivative of the dilaton field. Thus, the power spectra computed by matching only the reduced amplitude perturbations is different from that obtained by matching the full metric perturbation.

In summary, the physical quantity to be studied is the full metric perturbation amplitude. Meanwhile, the reduced amplitude perturbation can be considered as a sometimes convenient mathematical reduction of the problem.

In the inflationary stage, we have the same equations for the reduced amplitude perturbation (4.13) and (4.14) that in the precedent section. Therefore, we will have the same expression for the reduced amplitude perturbation:

$$\psi_I \;=\; \mathcal{N}(\eta_I - \bar{\bar{\eta}})^{\frac{1}{2}}\mathcal{H}_\nu{}^{(1)}\left(k(\eta_I - \bar{\bar{\eta}})\right) \tag{5.54}$$

with

$$\nu \;=\; \pm\left(\frac{d+1}{2}q - \frac{1}{2}\right) \tag{5.55}$$

The reduced ψ and the total tensorial metric perturbation are related by eq.(4.15). Instead of ψ, we will match the metric perturbation amplitude Φ:

$$\Phi \;=\; \psi a^{-\frac{d-1}{2}} e^{\frac{\phi}{2}} \tag{5.56}$$

such that the metric perturbation is written as:

$$h_{\mu\nu} \;=\; e_{\mu\nu}(\mathbf{k})\;\Phi(\eta,\mathbf{k})\;e^{i\mathbf{k}.\mathbf{x}} \tag{5.57}$$

From eqs.(2.6), (3.5) and (5.47), we have:

$$\Phi_I \;=\; \mathcal{N}{\alpha_I}^{\frac{d+1}{2}} e^{\frac{\phi_I}{2}}\;(\eta_I-\bar{\bar{\eta}})^{-q\left(\frac{d+1}{2}\right)+\frac{1}{2}}{\mathcal{H}_\nu}^{(1)}\left(k(\eta_I-\bar{\bar{\eta}})\right) \tag{5.58}$$

In contrast to the precedent cases, we will take in the following the negative sign in the parameter ν (5.55):

$$\nu \;=\left(\frac{1}{2}\;-\;q\frac{d+1}{2}\right) \tag{5.59}$$

because with this definition, we can write the amplitude perturbation as:

$$\Phi_I \;=\; \mathcal{N}{\alpha_I}^{\frac{d+1}{2}} e^{\frac{\phi_I}{2}}\;(\eta_I-\bar{\bar{\eta}})^{\nu}{\mathcal{H}_\nu}^{(1)}\left(k(\eta_I-\bar{\bar{\eta}})\right) \tag{5.60}$$

where $\mathcal{N}$ is a normalization constant.Notice the parameter ν in the conformal time dependence.

On the other hand, in the radiation dominated stage, the reduced amplitude perturbation is obtained in a totally equivalent way to the precedent section. We recover for this reduced amplitude eq.(5.53):

$$\psi_{II} = i\mathcal{M}\sqrt{\frac{2}{\pi k}}\Big(\alpha e^{-ik(\bar{\eta}-\eta_1)}+\beta e^{ik(\bar{\eta}-\eta_1)}\Big) \tag{5.61}$$

since we have again the parameter $\mu=\pm\frac{1}{2}$.

The tensorial metric perturbation $h_{\mu\nu}$ is given by eq.(5.57) with the amplitude Φ given by:

$$\Phi_{II} = i\mathcal{M}\sqrt{\frac{2}{\pi k}}e^{\frac{\phi_{II}}{2}}{\alpha_{II}}^{-\frac{d-1}{2}}\bar{\eta}^{-r\left(\frac{d-1}{2}\right)}\Big(\alpha e^{-ik(\bar{\eta}-\eta_1)}+\beta e^{ik(\bar{\eta}-\eta_1)}\Big) \tag{5.62}$$

where $r=\frac{2}{d-1}$. In shorter form, introducing the notation:

$$\mathcal{F} = \mathcal{N}{\alpha_I}^{\frac{d+1}{2}}e^{\frac{\phi_I}{2}} \tag{5.63}$$

$$\mathcal{G} = i\sqrt{\frac{2}{\pi}}\,\mathcal{M}\,{\alpha_{II}}^{-\frac{d-1}{2}}e^{\frac{\phi_{II}}{2}} \quad , \tag{5.64}$$

we have:

$$\Phi_I = \mathcal{F}(\eta_I-\bar{\bar{\eta}})^{\nu}{\mathcal{H}_\nu}^{(1)}\left(k(\eta_I-\bar{\bar{\eta}})\right) \tag{5.65}$$

$$\Phi_{II} = \mathcal{G}k^{-\frac{1}{2}}\bar{\eta}^{-1}\Big(\alpha e^{-ik(\bar{\eta}-\eta_1)}+\beta e^{ik(\bar{\eta}-\eta_1)}\Big) \tag{5.66}$$

where ν is given by eq.(5.55).

As indicated above, in this case we will compute the Bogoliubov's Coefficients by matching the total amplitude of tensorial metric perturbation Φ. Thus, we have to solve the equations:

$$\Phi_I(\bar{\bar{\eta}} = \eta_1) = \Phi_{II}(\bar{\eta} = \eta_1) \tag{5.67}$$

$$\left.\frac{d\Phi_I}{d\bar{\bar{\eta}}}\right|_{\bar{\bar{\eta}} = \eta_1} = \left.\frac{d\Phi_{II}}{d\bar{\eta}}\right|_{\bar{\eta} = \eta_1} \tag{5.68}$$

which yield:

$$\mathcal{F}\left(\frac{q}{r}\eta_1\right)^{\nu}\mathcal{H}_{\nu}{}^{(1)}\left(k\frac{q}{r}\eta_1\right) = \mathcal{G}k^{-\frac{1}{2}}\eta_1{}^{-1}(\alpha+\beta) \tag{5.69}$$

$$\mathcal{F}\nu\left(\frac{q}{r}\eta_1\right)^{\nu-1}\mathcal{H}_{\nu}{}^{(1)}\left(k\frac{q}{r}\eta_1\right) + k\mathcal{F}\left(\frac{q}{r}\eta_1\right)^{\nu}\mathcal{H}'_{\nu}{}^{(1)}\left(k\frac{q}{r}\eta_1\right) = \tag{5.70}$$

$$= \mathcal{G}k^{-\frac{1}{2}}\eta_1{}^{-2}(\beta(1-ik\eta_1)+\alpha(1+ik\eta_1))$$

By solving this system, we have:

$$\alpha = -i\frac{\mathcal{F}}{2\mathcal{G}}\eta_1{}^{\nu+1}\left(\frac{q}{r}\right)^{\nu}k^{\frac{1}{2}}\left[\left(\frac{\nu}{X} - \frac{\frac{q}{r}}{X} + i\right)\mathcal{H}_{\nu}{}^{(1)}(X) + \mathcal{H}'_{\nu}{}^{(1)}(X)\right] \tag{5.71}$$

$$\beta = i\frac{\mathcal{F}}{2\mathcal{G}}\eta_1{}^{\nu+1}\left(\frac{q}{r}\right)^{\nu}k^{\frac{1}{2}}\left[\left(\frac{\nu}{X} - \frac{\frac{q}{r}}{X} - i\right)\mathcal{H}_{\nu}{}^{(1)}(X) + \mathcal{H}'_{\nu}{}^{(1)}(X)\right] \tag{5.72}$$

where X is given by eq.(5.19).

Finally, after computation and using the properties of Hankel functions and their wronskian (See [27] and [28]), we have the following expressions for the square moduli:

$$\begin{aligned}\mid\alpha\mid^2 = \frac{\mid\mathcal{F}\mid^2}{\mid\mathcal{G}\mid^2}\,\eta_1{}^{2\nu+2}\left(\frac{q}{r}\right)^{2\nu}\frac{k}{4}\,&\left[\left(1+\frac{\left(\frac{q}{r}\right)^2}{X^2}\right)\mathcal{H}_{\nu}{}^{(1)}(X)\mathcal{H}_{\nu}{}^{(2)}(X) + \right.\\ &+\mathcal{H}_{\nu-1}{}^{(1)}(X)\mathcal{H}_{\nu-1}{}^{(2)}(X)\\ &\left.-\;2\frac{\frac{q}{r}}{X}\mathcal{H}_{\nu}{}^{(1)}(X)\mathcal{H}_{\nu-1}{}^{(2)}(X) - i\frac{4}{\pi}\frac{\frac{q}{r}}{X^2} + \frac{4}{\pi X}\right]\end{aligned} \tag{5.73}$$

$$\begin{aligned}\mid\beta\mid^2 = \frac{\mid\mathcal{F}\mid^2}{\mid\mathcal{G}\mid^2}\,\eta_1{}^{2\nu+2}\left(\frac{q}{r}\right)^{2\nu}\frac{k}{4}\,&\left[\left(1+\frac{\left(\frac{q}{r}\right)^2}{X^2}\right)\mathcal{H}_{\nu}{}^{(1)}(X)\mathcal{H}_{\nu}{}^{(2)}(X) + \right.\\ &+\mathcal{H}_{\nu-1}{}^{(1)}(X)\mathcal{H}_{\nu-1}{}^{(2)}(X)\\ &\left.-\;2\frac{\frac{q}{r}}{X}\mathcal{H}_{\nu}{}^{(1)}(X)\mathcal{H}_{\nu-1}{}^{(2)}(X) - i\frac{4}{\pi}\frac{\frac{q}{r}}{X^2} - \frac{4}{\pi X}\right]\end{aligned} \tag{5.74}$$

From expressions (5.73) and (5.74) is easy to see:

$$| \alpha |^2 - | \beta |^2 \;=\; \frac{| \mathcal{F} |^2}{| \mathcal{G} |^2} \, {\eta_1}^{2\nu+2} \left(\frac{q}{r}\right)^{2\nu} \frac{k}{4} \left[\frac{8}{\pi \, k \, \frac{q}{r} \, \eta_1}\right] \tag{5.75}$$

which by applying the normalization condition of Bogoliubov's Coefficients eq.(5.31), gives:

$$\frac{| \mathcal{F} |^2}{| \mathcal{G} |^2} {\eta_1}^{2\nu+1} \left(\frac{q}{r}\right)^{2\nu-1} \frac{2}{\pi} \;=\; 1 \tag{5.76}$$

Now, we can translate this condition over the usual normalization constants $\mathcal{N}$ and $\mathcal{M}$. From the definitions (5.63) and (5.64):

$$\frac{| \mathcal{F} |^2}{| \mathcal{G} |^2} \;=\; \frac{| \mathcal{N} |^2}{| \mathcal{M} |^2} \, \frac{\pi}{2} \, \frac{e^{\phi_I}}{e^{\phi_{II}}} \, \frac{{\alpha_I}^{d+1}}{{\alpha_{II}}^{1-d}}$$

From the matching properties of the scale factor in conformal time eq.(3.6), we have:

$$\frac{\alpha_I}{\alpha_{II}} \;=\; \left(\frac{q}{r}\right)^{q} {\eta_1}^{r+q} \tag{5.77}$$

And from eqs.(3.13) and (3.12):

$$\frac{e^{\phi_I}}{e^{\phi_{II}}} \;=\; {\alpha_{II}}^{-2d} \, {\eta_1}^{-2dr} \tag{5.78}$$

With eqs.(5.77) and (5.78) and the ν value eq.(5.59) we obtain:

$$\frac{| \mathcal{F} |^2}{| \mathcal{G} |^2} \;=\; \frac{| \mathcal{N} |^2}{| \mathcal{M} |^2} \, \frac{\pi}{2} \left(\frac{q}{r}\right)^{1-2\nu} {\eta_1}^{1-2\nu+r(1-d)} \tag{5.79}$$

Since $r = \frac{2}{d-1}$, eqs.(5.76) and (5.79) yield:

$$\frac{| \mathcal{N} |^2}{| \mathcal{M} |^2} \;=\; {\eta_1}^{-2-r(1-d)} \;=\; 1 \tag{5.80}$$

That is, the normalization factors $\mathcal{N}$ and $\mathcal{M}$ differ only in a phase. We can eliminate the normalization constants in the expressions for the Bogoliubov's Coefficients. From eq.(5.76), the coefficient multiplying the square brackets expression in eqs.(5.73) and (5.74) is exactly $\frac{\pi}{8}\frac{q}{r}k\eta_1$. Then, the final expressions for the α and β coefficients are:

$$\begin{aligned} | \alpha |^2 = \frac{\pi}{8} \, X \; \Bigg[& \left(1 + \frac{\left(\frac{q}{r}\right)^2}{X^2}\right) {\mathcal{H}_\nu}^{(1)}(X) {\mathcal{H}_\nu}^{(2)}(X) + \\ & + {\mathcal{H}_{\nu-1}}^{(1)}(X) {\mathcal{H}_{\nu-1}}^{(2)}(X) \\ & - \; 2\frac{\frac{q}{r}}{X} {\mathcal{H}_\nu}^{(1)}(X) {\mathcal{H}_{\nu-1}}^{(2)}(X) - i\frac{4}{\pi}\frac{\frac{q}{r}}{X^2} + \frac{4}{\pi X} \Bigg] \end{aligned} \tag{5.81}$$

$$
\begin{aligned}
|\,\beta\,|^2 = \frac{\pi}{8}\; X \;&\left[\left(1+\frac{\left(\frac{q}{r}\right)^2}{X^2}\right) \mathcal{H}_{\nu}{}^{(1)}(X)\mathcal{H}_{\nu}{}^{(2)}(X) + \right. \\
&+ \mathcal{H}_{\nu-1}{}^{(1)}(X)\mathcal{H}_{\nu-1}{}^{(2)}(X) \\
&\left. -\;\; 2\frac{\frac{q}{r}}{X}\mathcal{H}_{\nu}{}^{(1)}(X)\mathcal{H}_{\nu-1}{}^{(2)}(X) - i\frac{4}{\pi}\frac{\frac{q}{r}}{X^2} - \frac{4}{\pi X}\right]
\end{aligned}
\tag{5.82}
$$

5.3.1 The Power Spectrum in the Full Dilaton Case

Using eq.(4.29) and (5.82), we have for the power spectrum the next expression:

$$
\begin{aligned}
P(\omega)d\omega = \frac{\hbar}{8\pi c^3}\omega^2\frac{\left(\frac{q}{r}\right)^2}{S}d\omega \;&\left[\left(1+\frac{(\omega\;S)^2}{\left(\frac{q}{r}\right)^2}\right) \mathcal{H}_{\nu}{}^{(1)}(\omega S)\mathcal{H}_{\nu}{}^{(2)}(\omega S) + \right. \\
&+ \frac{(\omega\;S)^2}{\left(\frac{q}{r}\right)^2}\mathcal{H}_{\nu-1}{}^{(1)}(\omega S)\mathcal{H}_{\nu-1}{}^{(2)}(\omega S) \;+ \\
&\left. -\;\; 2\frac{\omega S}{\frac{q}{r}}\mathcal{H}_{\nu}{}^{(1)}(\omega S)\mathcal{H}_{\nu-1}{}^{(2)}(\omega S) - i\frac{4}{\pi\frac{q}{r}} - \frac{4}{\pi}\frac{\omega S}{\left(\frac{q}{r}\right)^2}\right]
\end{aligned}
\tag{5.83}
$$

The fraction of critical energy density takes the expression:

$$
\begin{aligned}
\Omega_{GW} = \frac{\hbar\;G}{3{H_0}^2c^5}\;\frac{\left(\frac{q}{r}\right)^2}{S}\;\omega^3 \;&\left[\left(1+\frac{(\omega\;S)^2}{\left(\frac{q}{r}\right)^2}\right) \mathcal{H}_{\nu}{}^{(1)}(\omega S)\mathcal{H}_{\nu}{}^{(2)}(\omega S) + \right. \\
&+ \frac{(\omega\;S)^2}{\left(\frac{q}{r}\right)^2}\mathcal{H}_{\nu-1}{}^{(1)}(\omega S)\mathcal{H}_{\nu-1}{}^{(2)}(\omega S) \;+ \\
&\left. -\;\; 2\frac{\omega S}{\frac{q}{r}}\mathcal{H}_{\nu}{}^{(1)}(\omega S)\mathcal{H}_{\nu-1}{}^{(2)}(\omega S) - i\frac{4}{\pi\frac{q}{r}} - \frac{4}{\pi}\frac{\omega S}{\left(\frac{q}{r}\right)^2}\right]
\end{aligned}
\tag{5.84}
$$

We compute for the three-dimensional case the values of the coefficients in front of the square brackets in eqs.(5.83) and (5.84). These values differ from our previous computations on (5.39) and (5.46) in a factor $\left(\frac{\frac{q}{r}}{\nu-\frac{1}{2}}\right)^2$. Thus, in the full dilaton case, we have:

$$
\frac{\hbar}{8\pi c^3}\;\frac{\left(\frac{q}{r}\right)^2}{S}\;\sim\;4.008\;10^{-55}\;\frac{\text{erg seg}^3}{\text{cm}^3} \tag{5.85}
$$

$$
\frac{\hbar\;G}{3{H_0}^2c^5}\;\frac{\left(\frac{q}{r}\right)^2}{S}\;\sim\;6.220\;10^{-47}\;\text{seg}^{-3} \tag{5.86}
$$

In the Tables (3) and (4), we show the different power spectra and fraction of critical energy density for a single cosmological scale factor, but in the three different roles for the dilaton. As can be seen, there is a notorious difference between the three different treatments of the dilaton role for a single cosmological scale factor description. First, the parameter ν, characterizing the inflationary expansion and the power spectrum, is different in each case. The sign of this parameter is only ensured in the full dilaton treatment, by relating it with the form of the metric amplitude perturbation. Second one, the coefficients in front of the square brackets are different. Within the reduced amplitude treatment (no dilaton and partial dilaton cases), these coefficients appear related with the quantity $\left(\nu - \frac{1}{2}\right)$, meanwhile for the total metric amplitude $h_{\nu\mu}$ matching (full dilaton case), the coefficients have the expression $\left(\frac{q}{r}\right)$.

It must be observed that the No Dilaton Case is coherently described also by the expressions corresponding to the Full Dilaton Case. In fact, in the first one, the next relation holds:

$$\nu - \frac{1}{2} = \frac{q}{r}$$

(this relation is not satisfied by the partial dilaton role case, which must be described by the specific expressions above indicated).

These observations enable us to work with general expressions for the stochastic background of gravitational waves, given by the corresponding ones of the full dilaton case eqs.(5.83) and (5.84). The partial dilaton and no dilaton cases can be simply obtained from these expressions by substituting the factors $\frac{q}{r}$ by $\left(\nu - \frac{1}{2}\right)$ with the proper value of parameter ν for each one.

Case	par. inflationary ν gral.	$d=3$	$(\nu-1)$	*Power Spectrum* $P(\omega)d\omega$
No Dilaton	$\pm\left(\frac{d-1}{2}q+\frac{1}{2}\right)$	$\frac{5}{6}$	$-\frac{1}{6}$	$\frac{\hbar}{8\pi c^3}\omega^2\frac{\left(\nu-\frac{1}{2}\right)^2}{S}d\omega \left[\left(1+\frac{(\omega S)^2}{\left(\nu-\frac{1}{2}\right)^2}\right)\mathcal{H}_{\nu}{}^{(1)}(\omega S)\mathcal{H}_{\nu}{}^{(2)}(\omega S) + \frac{(\omega S)^2}{\left(\nu-\frac{1}{2}\right)^2}\mathcal{H}_{\nu-1}{}^{(1)}(\omega S)\mathcal{H}_{\nu-1}{}^{(2)}(\omega S) - \frac{2\,\omega S}{\left(\nu-\frac{1}{2}\right)}\mathcal{H}_{\nu}{}^{(1)}(\omega S)\mathcal{H}_{\nu-1}{}^{(2)}(\omega S) - \frac{4\,\omega S}{\pi\left(\nu-\frac{1}{2}\right)^2} - i\frac{4}{\pi\left(\nu-\frac{1}{2}\right)}\right]$
Partial Dilaton	$\pm\left(\frac{d+1}{2}q-\frac{1}{2}\right)$	$\frac{1}{6}$	$-\frac{5}{6}$	
Full Dilaton	$\left(\frac{1}{2}\ -\ q\frac{d+1}{2}\right)$	$-\frac{1}{6}$	$-\frac{7}{6}$	$\frac{\hbar}{8\pi c^3}\omega^2\frac{\left(\frac{q}{r}\right)^2}{S}d\omega \left[\left(1+\frac{(\omega\ S)^2}{\left(\frac{q}{r}\right)^2}\right)\mathcal{H}_{\nu}{}^{(1)}(\omega S)\mathcal{H}_{\nu}{}^{(2)}(\omega S) + \frac{(\omega\ S)^2}{\left(\frac{q}{r}\right)^2}\mathcal{H}_{\nu-1}{}^{(1)}(\omega S)\mathcal{H}_{\nu-1}{}^{(2)}(\omega S) - 2\frac{\omega S}{\frac{q}{r}}\mathcal{H}_{\nu}{}^{(1)}(\omega S)\mathcal{H}_{\nu-1}{}^{(2)}(\omega S) - i\frac{4}{\pi\frac{q}{r}} - \frac{4}{\pi}\frac{\omega S}{\left(\frac{q}{r}\right)^2}\right]$

Table 3: The Dilaton Role in the Power Spectrum in String Cosmology

Comparative Table of the Gravitational Waves Power Spectrum obtained in our String Cosmology Model. Differences have oughted to the weight of the dilaton in each case. The differences between the three cases turn out in the value of the ν parameter and the coefficients in the power spectrum expresssions.

Case	par. inflationary ν gral.	$d=3$	$(\nu-1)$	*Contribution to Energy Density* Ω_{GW}
No Dilaton	$\pm\left(\frac{d-1}{2}q+\frac{1}{2}\right)$	$\frac{5}{6}$	$-\frac{1}{6}$	$\frac{\hbar\, G}{3{H_0}^2c^5}\,\frac{\left(\nu-\frac{1}{2}\right)^2}{S}\,\omega^3 \Big[\Big(1+\frac{(\omega S)^2}{\left(\nu-\frac{1}{2}\right)^2}\Big){\mathcal{H}_\nu}^{(1)}(\omega S){\mathcal{H}_\nu}^{(2)}(\omega S) + \frac{(\omega S)^2}{\left(\nu-\frac{1}{2}\right)^2}\,{\mathcal{H}_{\nu-1}}^{(1)}(\omega S){\mathcal{H}_{\nu-1}}^{(2)}(\omega S) - \frac{2\,\omega S}{\left(\nu-\frac{1}{2}\right)}{\mathcal{H}_\nu}^{(1)}(\omega S){\mathcal{H}_{\nu-1}}^{(2)}(\omega S) - \frac{4\,\omega S}{\pi\left(\nu-\frac{1}{2}\right)^2} - i\frac{4}{\pi\left(\nu-\frac{1}{2}\right)}\Big]$
Partial Dilaton	$\pm\left(\frac{d+1}{2}q-\frac{1}{2}\right)$	$\frac{1}{6}$	$-\frac{5}{6}$	
Full Dilaton	$\left(\frac{1}{2}\ -\ q\frac{d+1}{2}\right)$	$-\frac{1}{6}$	$-\frac{7}{6}$	$\frac{\hbar\, G}{3{H_0}^2c^5}\,\frac{\left(\frac{q}{r}\right)^2}{S}\,\omega^3 \Big[\Big(1+\frac{(\omega\ S)^2}{\left(\frac{q}{r}\right)^2}\Big){\mathcal{H}_\nu}^{(1)}(\omega S){\mathcal{H}_\nu}^{(2)}(\omega S) + \frac{(\omega\ S)^2}{\left(\frac{q}{r}\right)^2}{\mathcal{H}_{\nu-1}}^{(1)}(\omega S){\mathcal{H}_{\nu-1}}^{(2)}(\omega S) - 2\frac{\omega S}{\frac{q}{r}}{\mathcal{H}_\nu}^{(1)}(\omega S){\mathcal{H}_{\nu-1}}^{(2)}(\omega S) - i\frac{4}{\pi\frac{q}{r}} - \frac{4}{\pi}\frac{\omega S}{\left(\frac{q}{r}\right)^2}\Big]$

Table 4: Contribution to Energy Density with different Dilaton Role in String Cosmology

Comparative Table of the Fraction of the Energy Density Ω_{GW} contained in Gravitational Wave Backgrounds produced in our String Cosmology Model. Again, the differences at the value of ν parameter and have oughted to the weight of the dilaton role in each case. Signatures of this are the value of the ν parameter and the coefficients in the power spectrum expresssions.

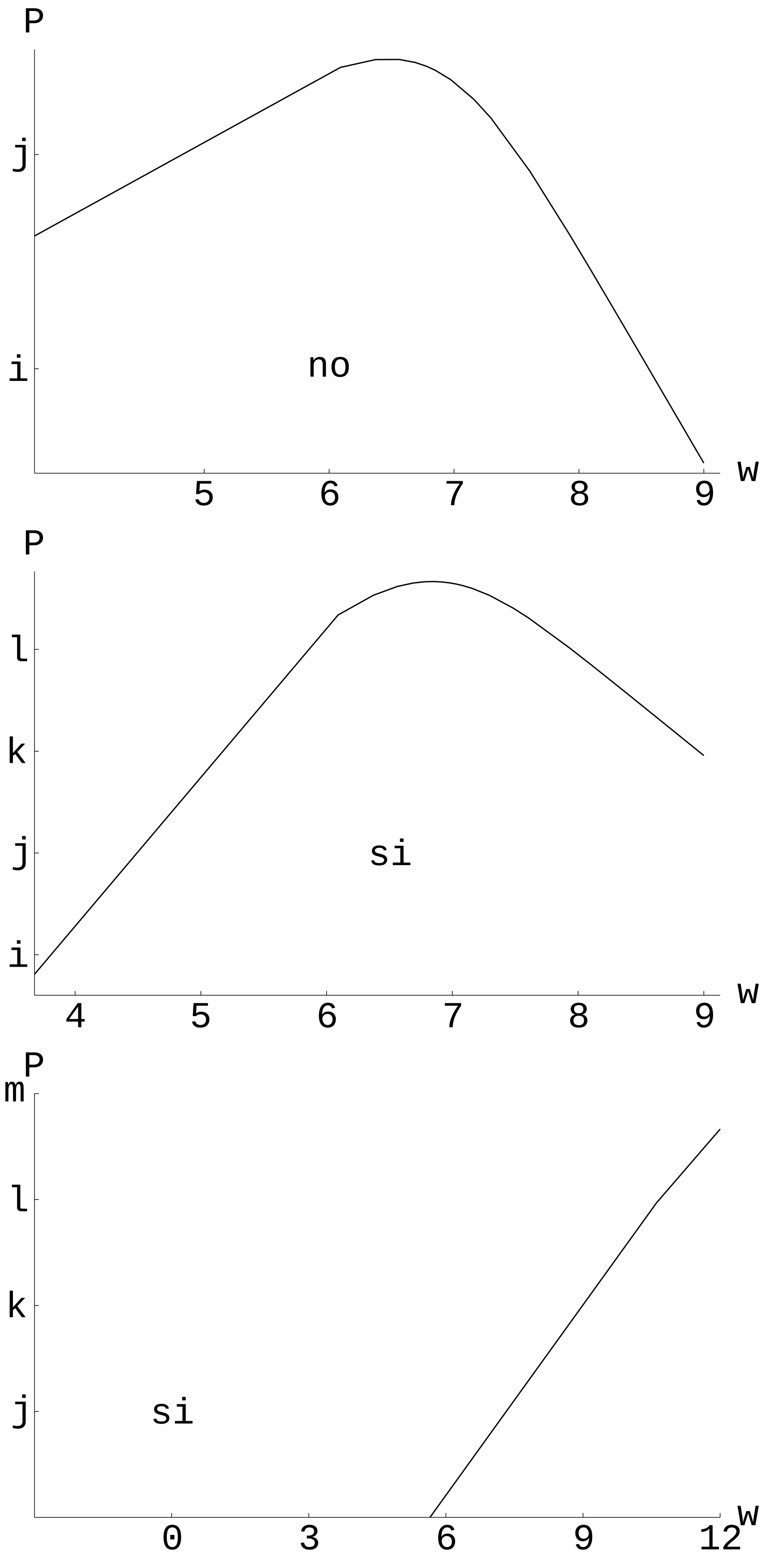

Figure 1: Power spectrum for No Dilaton, partial Dilaton and Full Dilaton Case

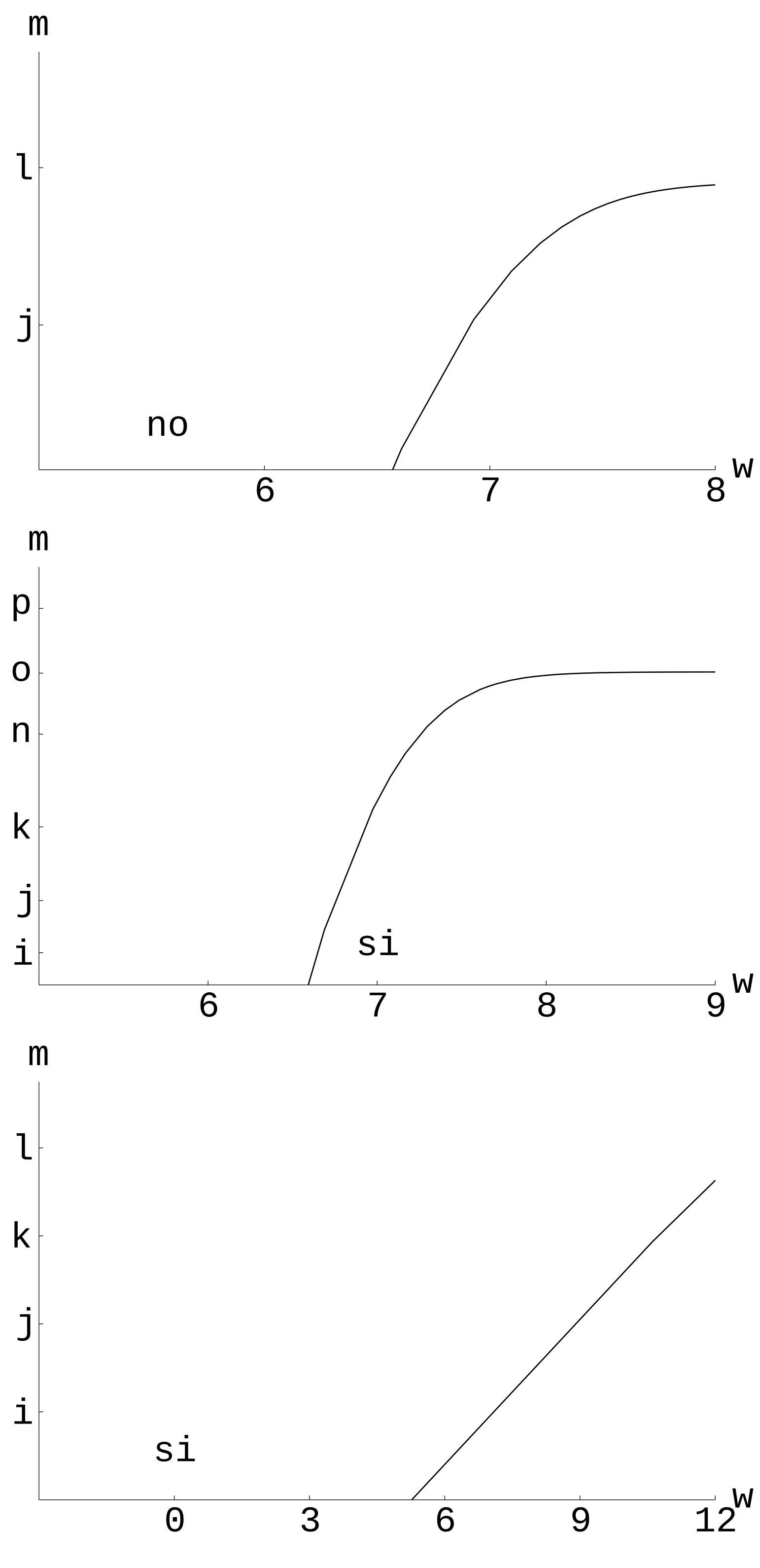

Figure 2: Contribution to Energy Density for the No Dilaton, partial Dilaton and Full Dilaton Case

6 Asymptotic Behaviours of the Gravitational Wave Background

We have computed the analytical expressions for $P(\omega)d\omega$ and Ω_{GW} at high and low frequencies. Both regimes correspond to very large ($X >> 1$) and very small ($X << 1$) arguments of Hankel functions, respectively. We consider the three regimes: low frequencies for $\omega << \omega_x$ ($X << 1$), medium frequencies for $\omega \sim 1$ (exact analytical expressions (5.38),(5.45),(5.83),(5.84)) and high frequencies for $\omega >> \omega_x$ ($X >> 1$) where ω_x is the characteristic frequency (eq.(5.36)) with order of Mhz for the scale factor evolution considered.

The exact analytical form of the power spectrum, eqs.(5.38) and (5.83) requires a careful algebraic treatment in order to compute their asymptotic behaviours. Attention must be put both in the range of validity as on the order of expansions used. Computation is quite long, thus we omit details here and give only the leader orders.

6.1 High frequencies

The High Frequency Asymptotic expressions for the No Dilaton (ND) and Partial Dilaton (PD) Cases are formally equivalent. The leader orders for the power spectrum and contribution to energy density can be written respectively as:

$$P(\omega)d\omega|_{ND,PD} \sim \frac{\hbar}{16\pi^2 c^3}\frac{\left(\nu-\frac{1}{2}\right)^2\left(\nu+\frac{1}{2}\right)^2}{S^4}\omega^{-1}+O(\omega^{-3}) \tag{6.1}$$

$$\Omega_{GW}|_{ND,PD} \sim \frac{\hbar\, G}{6{H_0}^2\pi c^5}\frac{\left(\nu-\frac{1}{2}\right)^2\left(\nu+\frac{1}{2}\right)^2}{S^4}+O(\omega^{-2}) \tag{6.2}$$

The Full Dilaton (FD) Case presents different expressions for the High Asymptotic Behaviour:

$$P(\omega)d\omega|_{FD} \sim \frac{\hbar}{4\pi^2 c^3}\frac{\left(\frac{q}{r}\right)^2}{S^2}\omega\left(\frac{\nu-\frac{1}{2}}{\left(\frac{q}{r}\right)}-1\right)^2+O(\omega^{-1}) \tag{6.3}$$

$$\Omega_{GW}|_{FD} \sim \frac{2\hbar\, G}{3{H_0}^2\pi c^5}\frac{\left(\frac{q}{r}\right)^2}{S^2}\omega^2\left(\frac{\nu-\frac{1}{2}}{\left(\frac{q}{r}\right)}-1\right)^2+O(\omega^{0}) \tag{6.4}$$

Notice that leader order terms have fixed dependences, independent of number of spatial dimensions or parameters of the solutions. In fact, $P(\omega)d\omega$ in both No Dilaton and Partial Dilaton cases present a convergent behaviour at high frequencies as $\sim \omega^{-1}$. In contrast, the Full Dilaton case is divergent in this regime as $\sim \omega^1$. Ω_{GW} in the No Dilaton and Partial Dilaton cases reachs asymptotically a constant value. In the Full Dilaton case Ω_{Gw} diverges as ω^2.

6.2 Low frequencies

Opposite to the high frequencies case, leader orders for low frequencies depend explicitly from the parameter ν. As a consequence, each case presents a particular expression for their asymptotic behaviours. To leader order uniquely, the No Dilaton case and the Partial Dilaton case have the same formal expressions:

$$P(\omega)d\omega|_{ND,PD} \sim \frac{\hbar}{8\pi c^3}\frac{\left(\nu-\frac{1}{2}\right)^2}{S^3}d\omega\frac{2^{2\nu}}{\pi^2}\Gamma(\nu)^2(\omega S)^{2-2\nu} \ + \ O((\omega S)^2) \tag{6.5}$$

$$\Omega_{GW}|_{ND,PD} \sim \frac{\hbar\, G}{3{H_0}^2c^5}\frac{\left(\nu-\frac{1}{2}\right)^2}{S^4}\frac{2^{2\nu}}{\pi^2}\Gamma(\nu)^2(\omega S)^{3-2\nu} \ + \ O((\omega S)^3) \tag{6.6}$$

In the three-dimensional case, $\nu = \frac{5}{6}$ for the (ND) case and $\nu = \frac{1}{6}$ for the (PD) case. We have at very low frequencies $P(\omega)d\omega|_{ND} \sim \omega^{\frac{1}{3}}$ and $\Omega_{GW}|_{ND} \sim \omega^{\frac{4}{3}}$, $P(\omega)d\omega|_{PD} \sim \omega^{\frac{5}{3}}$ and $\Omega_{GW}|_{PD} \sim \omega^{\frac{8}{3}}$.

In the Full Dilaton Case the low frequencies behaviour is:

$$P(\omega)d\omega|_{FD} \sim \frac{\hbar}{8\pi c^3}\frac{\left(\frac{q}{r}\right)^2}{S^3}d\omega\frac{\Gamma(1-\nu)^2}{2^{2\nu}\pi^2}\left(\frac{2}{\frac{q}{r}}-\frac{1}{\nu}\right)^2(\omega S)^{2+2\nu} + O((\omega S)^2) \tag{6.7}$$

$$\Omega_{GW}|_{FD} \sim \frac{\hbar\, G}{3{H_0}^2c^5}\frac{\left(\frac{q}{r}\right)^2}{S^4}\frac{\Gamma(1-\nu)^2}{2^{2\nu}\pi^2}\left(\frac{2}{\frac{q}{r}}-\frac{1}{\nu}\right)^2(\omega S)^{3+2\nu} + O((\omega S)^3) \tag{6.8}$$

The leader order has the same behaviour $P(\omega)d\omega \sim \omega^{\frac{5}{3}}$ and $\Omega_{GW} \sim \omega^{\frac{8}{3}}$ as in the Partial Dilaton Case, but the coefficient is different.

The leader order at very asymptotic behaviours for all cases is summarized in table (5).

Case		$P(\omega)d\omega$		Ω_{GW}	
		Low Frq.	**High Frq.**	**Low Frq.**	**High Frq.**
String	No Dilaton	$\sim \omega^{\frac{1}{3}}$	$\sim \omega^{-1}$	$\sim \omega^{\frac{4}{3}}$	cte.
Driven	partial Dil.	$\sim \omega^{\frac{5}{3}}$	$\sim \omega^{-1}$	$\sim \omega^{\frac{8}{3}}$	cte.
	Full Dil.	$\sim \omega^{\frac{5}{3}}$	$\sim \omega$	$\sim \omega^{\frac{8}{3}}$	$\sim \omega^2$

Table 5: Asymptotic Behaviours of Gravitational Wave Power Spectrum and Contribution to Energy Density.

The dominant dependence for very low and very high frequences are summarized. The values are refered to the three spatial dimensional case.

7 Discussion and Conclusions

We have studied the production of a primordial stochastic gravitational wave background in a cosmological model fully extracted in the context of selfconsistent string cosmology. As seen in Section II, suitable descriptions for inflationary, radiation dominated and matter dominated stages can be extracted from effective string theory in which strings propagating in the curved backgrounds are the classical sources of matter and drive selfconsistently the cosmological background. The equation of state provided by the dynamics of string matter naturally generates in its evolution the background in the three stages.

No exact description of the transitions among stages can be extracted in the framework of this effective treatment. Higher order corrections in low energy effective equations or more detailed study on the evolution of the gas of strings would be neccessaries in order to handle such subject.

This lack of full understanding of the transition dynamics is an usual limitation when primordial metric perturbations are computed. The main consequence is the loss of predictibility. Usual gravitational wave results depend on parameters of the cosmological model which are not always well known ([14],[17]), or whose physical meaning must be recovered after computation ([16]), or where the not considered transition dynamics must be recalled in order to overcome divergences ([4], [18]). Usual arguments dealing with horizon exit and reentry ([5],[6],[8]) require a suitable cosmological description in order to be translated univocally into current measurement abilities.

We faced this problem before to compute the metric perturbations. We elaborate a cosmological minimal model of evolution for the scale factor with satisfactory sudden and continuous transitions, provided it makes use of particular descriptive cosmic time variables at each stage (see eqs.(3.2)). In this way, we merge our lack of knowledge about real transitions by modelizing them in a descriptive scale factor expressed in suitable cosmic time variables. This description is linked with the minimal observational Universe information (standard values for transition times and scale factor ratii reached in each stage). It provides an equivalent evolution for the scale factor, but evades the undetermined region of transitions without introduction of free parameters or loss of predictibility. Unless the details imprinted by dynamics transitions, in our case modelized as sudden and continuous, our computation is expected equivalent to one made on a full physical model running on cosmic time.

We have also imposed continuity and smothness on the conformal time description, by constructing this on the descriptive variables (see eqs.(3.5)). Continuity of scale factor both in cosmic time and conformal time description is a feature not always present in literature (see [24]). In our case, the variables constructed enable us to maintain the linking with the observational Universe information. This model seems us more appropiated in order to perform primordial metric perturbations computations, since no effects oughted to discontinuity in the scale factor of the metric are introduced. Not so perfect is the treatment of the dilaton field, whose evolution can be treated solely continuous but not smooth. Further study would be needed in order to valutate this effect.

The only gravitational waves we have computed are those produced as amplification by the evolution of background of tensorial metric perturbations. We do not account here another possible mechanisms of generation neither amplification of another metric perturbations.

The variable X in the power spectrum and the proper frequency ω are related in a way totally determined by the description of the cosmological scale factor evolution, see eq.(5.34). The factor relating them depends of the expansion ratii, the exit time of inflationary epoch and the coefficients of expansion inflationary and current epochs. Being all them fixed in our cosmological background linked to the observational Universe, no free parameters are introduced at this level. No one of remaining unknown parameters, like global scale factor $\bar{A_{II}}$, appears on the results of our computation. Differently from almost all string cosmology computations in literature, firm predictions on precise frequencies ranges can be extracted in our case.

In this way, we have computed exact, fully predictive and free-parameter expressions for the power spectrum $P(\omega)d\omega$ and contribution to energy density Ω_{GW} of the primordial gravitational waves background generated in the transition among the inflationary and radiation dominated stages. We have not considered the contribution given by the radiation dominated-matter dominated transition in this study. The gravitational wave contribution due to this transition is expected to be neglectelly small, as compared to the first transition. It would be expected the second transition having a role only on the low frequencies regime [4], not so important in anycase for our results.

7.1 The Dilaton Role

We have obtained drastic differences in the stochastic background of gravitational waves produced in the same scale factor evolution by considering differents degrees of the role played by the dilaton. The simplest case, without account of the effect of the dilaton neither on the perturbation equation nor on the amplitude perturbation. The second case, a partial account, with the proper perturbation equation but still matching the reduced amplitude perturbation. The lastest case, a full account by working with the total tensorial amplitude perturbation and perturbation equation.

The background of gravitational waves comes characterized in their shape by the parameter ν, which depends of the inflationary description, the inflation-radiation dominated transition (the only on which we are computing gravitational wave production) and the role played by the dilaton. The expressions for ν have been found in the three cases (eqs.(5.4),(5.49) and (5.59)). We obtain an exact expression for the power spectrum and energy density contribution (eqs.(5.38) and (5.45)) in terms of Hankel functions of order ν, formally equal in the No Dilaton and partial Dilaton cases. Differences among them are oughted to ν differences. The formal expressions in the full dilaton case are different (eqs.(5.83),(5.84)) both in parameter ν as in coefficients involved.

The low frequency and high frequency asymptotic regimes have been discussed. In the No Dilaton case, asymptotic behaviours for power spectrum are both vanishing at low and high frequencies as $\omega^{\frac{1}{3}}$ and ω^{-1} respectively. This gives a gravitational wave contribution to the energy density asymptotically constant at high frequencies of magnitude $\Omega_{GW} \sim 10^{-26}$. There is a slope change that produces a peak in the power spectra around a characteristic frequency totally determined by the model of $\omega_x \sim 1.48$ Mhz.

The Partial Dilaton case introduces the effect of the dilaton exclusively in the tensorial perturbation equation (which is not longer equivalent to the massless real scalar field propagation equation [3],[9]), but not on the perturbation itself. The general characteristic are very similar to the previous case. Both asymptotic regimes for $P(\omega)d\omega$ vanish again, but with dependences $\omega^{\frac{5}{3}}$ and ω^{-1}. The peak appears around the same characteristic frequency, with value one order of magnitude lower than in the No Dilaton case, as well as the asymptotic constant contribution to energy density.

In contrast, when the full dilaton role is accounted, general characteristics as well as orders of magnitude of the spectrum are drastically modified. It has similar values for the frequencies below the Mhz, with power spectrum vanishing again as $\omega^{\frac{5}{3}}$. For high frequencies, in opposition to the former cases, both $P(\omega)d\omega$ and Ω_{GW} are increasing at high frequencies. For $P(\omega)d\omega$, an asymptotic divergent behaviour proportional to ω is found. It gives values much higher than the no dilaton and partial dilaton cases. The contribution to Ω_{GW} is equally divergent at high frequencies as ω^2. The change of slope is less visible and no clear peaks are found. The transition from the low frequency to the high frequency regime is slower than in the previous case and the full analytical expressions are needed on a wider range $10^6 \sim 10^9 Hz$. Comparative tables of these summarized results can be found in tables (3), (4) and (5). See also figs.(1) and (2).

Existence of an upper cutoff must be studied, whereas the Full Dilaton Case studied could be in contradiction with observational bounds as the current value of total energy density in critical units $\Omega \sim 1$. In that case, an end-point not predicted by the current minimal model considerations could be introduced in the spectrum as made in the literature [18]. Divergent high frequencies behaviour and introduction of an upper cutoff is an usual feature in the string cosmology contexts.

7.2 String and No-String Cosmologies

Among the spectra computed in string cosmology contexts, it must be distinguished between those computed in Brans-Dicke frames (that we compare with our Full Dilaton Case) and those computed following an usual quantum field theory way, that is, as our No Dilaton Case.

The shape of the spectra computed in string cosmology contexts are very similar. The principal features, as slope changings, are signal of the number of stages or transitions considered in the scale factor evolution. All the known cases, coherently treated in Brans-Dicke frames, presents an increasing dependence at high frequencies.

Notice that we are not handling with a "Pre-Big Bang" scenario [14]. Our String Driven Cosmological Background runs on positive values for the proper cosmic time t. It have been said [18] that Pre Big-Bang scenario predicts a power spectrum with a peak and an end-point. Our analysis enable us to identify these features with the Brans-Dicke framework where the low effective energy treatment is made, without necessity of a "Pre Big Bang" phase. In fact we have seen as the same Universe evolution and transitions have given different power spectra. In the full dilaton treatment, the transition among the inflationary inverse power stage and the radiation dominated stage gives an always increasing function for the number of particles $\mid \beta \mid^2$. It causes the high frequency range to be asymptotically divergent, feature that disappears completely in the No Dilaton or in the Partial dilaton treatments.

Similarly, a cosmological model involving successively a named Dilaton Driven stage plus an "String Phase" at nearly constant curvature, and a radiation dominated stage produces also a spectrum increasing with frequency ([16]). As made in the literature ([18]), an upper cutoff is placed on this increasing spectra by considering the frequency making $\mid \beta(\omega) \mid^2 = 1$ as the maximum one produced. This frequency is the one where one graviton per space phase unit volume and polarization is produced, and after which exponential suppression is supposed. These arguments leds to identification of the so called "end-point" that acts as "peak" of the spectrum oughted to the increasing power with frequency. The exact values at current time are functions of the undetermined parameters in the cosmological model there treated [16].

We conclude that whatever gravitational wave computation on inflationary stages of the type extracted in string cosmology, coherently made in the Brans-Dicke frame, must give an increasing spectrum. The peaks are produced by slope change and they are signal of the transitions in the dynamics of background evolution. We consider the Pre-Big Bang scenario do not predicts a peak, but it is supposed by defining a ω_1 such that $|\beta(\omega_1)|^2 = 1$. This proper frequency is computed at the beginning of radiation dominated stage, when the wave reenters the horizon and it must suffer a redshift at current time, expressable as function of unknown parameters of a "string phase". It acts as end-point because waves with $\omega > \omega_1$ are supposed exponentially suppressed. Since the spectrum was increasing with frequency, the same frequency ω_1 constitutes a maximun (peak).

If we use the same argument in order to fix an upper limit, our spectrum must be cutted at frequency $\omega_{max} \sim 3.85$ MHz where the power spectrum will have a value around $P(\omega)d\omega \sim 5.68\ 10^{-41}\frac{\mathrm{erg.s}}{\mathrm{cm}^3}$ and $\Omega_{GW} \sim 3.40\ 10^{-26}\rho_c$. These are the same order of magnitudes of the peaks atteinted on the No Dilaton and partial Dilaton Cases. No conflict with observational constraints look possible for such weak signals predicted. But this argument could be too naive, since in the practical way is equivalent to cancelate from the spectra the features introduced by the full dilaton role.

For instance, if we want to relaxe this condition and we ask us what would be the gravitational wave with maximum possible frequency, we can give at least a coherency condition. Now, let us ask if it is possible to produce a wave with period lower than the Planck time, that means with wavelenght lower than the Planck lenght scale. If that feature is possible, it is clear that our simplified quantum theory formalism would be unappropiated in order to compute such produced particles. We can fix the upper cutoff in the corresponding $\omega_P \sim 10^{44} Hz$ by considering that beyond it the power spectrum of gravitational waves must be, at least, different from elsewhere hitohere computed. At those values, both the No Dilaton and partial Dilaton case predicts completely negligible production, but not is the case for Full Dilaton Case. Its asymptotic divergent behaviour predicts enormous values for $P(\omega)d\omega$ and Ω_{GW}.

Obviously, we do not mean that these are the right predictions of string cosmology. Without doubt, the model here treated is excessively simple in order to handle properly the end-point of spectrum. But usual assumptions on upper exponential suppression could be more careful revisited, since a too naive treatment in this point can mask important features of predictions.

In relation with no-string inflationary cosmologies, the so called standard inflation is usually intended as a De Sitter stage. Notice that string driven inflationary stage describes an evolution with inverse power dependence. It must not be confused with usual power law, although our model can be said superinflationary too.

There is a radical difference among the string cosmology spectra and those obtained with an exponential inflationary expansion [4]. The divergence at low frequencies that it supposes is not found in our String Driven Cosmological Background The high frequencies behaviour are compatible with the No Dilaton Case (5.38), computed in a totally equivalent way.

If comparaison is made with the Full Dilaton Case, a totally different behaviour is found with respect to the obtained in De Sitter case. Notice that comparaison among string comology inflationary models and standard inflation means confrontate power and inverse power type laws with De Sitter exponential inflation, since until the moment no De Sitter type expansion have been coherently obtained in String Cosmology. This difference in the inflationary scale factor dynamics, together with the appropiated teatment of metric perturbations in each framework are the main causes of differences among the gravitational wave power spectra in both cases.

No explicit dependence on the beginning of the inflationary stage t_i has been found on the gravitational wave computation. In anycase, further study must be done in order to determine the influence on the power spectra if earliers inflation stages are considered. Gravitational wave production in frameworks with multistages inflation can be found in literature, both in the most classical cases ([26],[8],[9]) meanwhile in the string cosmology contexts there are various multistage models ([14],[16],[17]).

In summary, there are a variety of effects affecting the gravitational wave power spectra and justifiying the differences among our results and another string and no string cosmologies. The most notorious difference is oughted to the existence of a dilaton field and a metric evolving in a Brans-Dicke frame. Differences are found on the equation for the tensorial perturbation, and the amplitude of perturbation depends itself on the dilaton field. Our spectra here computed have showed clearly the successive effect obtained on the gravitational wave production by introducing the dilaton effect on the perturbation equation (partial Dilaton case) and on the perturbation itself (Full Dilaton case). From an unique evolution of the scale factor, we have seen as the dilaton role on the spectra consists in increasing the high frequencies regime, leading to a divergent behaviour. This is in agreement with almost string cosmology, Brans-Dicke computations. In our cases without full account of the dilaton role, we have spectra decreasing at higher frequencies, as such obtained in the literature [4] in the more standard frameworks.

There are many points that desserve further study in order to be clarified, either from the point of view of string cosmological backgrounds and transitions dynamics, either from those of gravitational wave computations in the string appropiated frameworks. Better treatments than here applied could take place in every phase of the problem, advising us against consider this procedure or its results as definitives. In anycase, we have proved here some aspects of the way predictions on observational consequences can be extracted from String Cosmology.

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